Composition of locally solid convergences

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Unless stated otherwise, the results are from:

- Bilokopytov, *Locally solid convergences and order continuity of positive operators*, 2023.
- Bilokopytov, Conradie, Troitsky, van der Walt, *Locally solid convergence structures*, 2024.
- Bilokopytov, *Composition of locally solid convergences*, 2025.

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Then, $\to$ on F defined by $f_\alpha \to f$ if $|f_\alpha - f| \to 0_F$ is a convergence on F.

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- σ*-order convergence*: same, but G is countable.
- *Relative uniform convergence:* $0_F \leq f_\alpha \xrightarrow{\mathrm{ru}} 0_F$ if there is $e \geq 0_F$ such that $\forall n \in \mathbb{N}$ there is α_n such that $f_\alpha \leq \frac{1}{n} e$, for all $\alpha \geq \alpha_n$.

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- *If θ is linear and completely metrizable, then $\eta_0\theta = \mathrm{ru}$.*

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Theorem 1

Let η_0 *be induced by the discrete topology on* F*. Then:*

- $f_\alpha \xrightarrow{\eta_0\theta} 0_F$ *iff there is a (decreasing)* $0_F \leq g_\beta \xrightarrow{\theta} 0_F$ *such that for every* β *there is* α_0 *such that* $f_\alpha \leq g_\beta$*, for all* $\alpha \geq \alpha_0$.
- $\eta_0\theta$ *is the strongest locally solid additive convergence on* F *in which every decreasing* θ*-null net is null.*
- *If* θ *is linear and completely metrizable, then* $\eta_0\theta = \mathrm{ru}$.
- *("Dini's theorem":) If* F *is a Banach lattice, then* $\eta_0\mathrm{aw} = \mathrm{ru}$.

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- *τ is minimal in this class;*
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Hence, if a minimal topology exists, it is unique, and is unbounded.

On the other hand, minimal Hausdorff locally solid (linear or additive) convergences always exist by Zorn's lemma.

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In this case $\mathrm{uo} =$ *"coordinatewise" convergence is the weakest.*

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THANK YOU!