Super-hedging problem in the Leland model with transaction costs: a backward-forward approach

COSA 2025, South of Africa

Emmanuel Lepinette
Joint work with Amal Omrani.

Ceremade CNRS, Paris-Dauphine University

Let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a stochastic basis.

A financial market model is composed (here) of one risk-free asset and one risky asset.

The (discounted) price of the risk-free asset is, at any time $t \geq 0$, $S_t^0 = 1 = (1+R)^t$, i.e. $R = 0$ w.l.o.g.

The discounted price of the risky asset is $\tilde{S}_t = S_t/S_t^0 = S_t$ where, at any time $t \geq 0$, S_t is $\mathcal{F}_t$-measurable.

A financial strategy is a stoc. proc. $\hat{\theta} = (\theta_t^0, \theta_t)_{t\in[0,T]}$, i.e. $\hat{\theta}_t$ is $\mathcal{F}_t$-measurable and θ_t^0, θ_t are the quantities of non-risky asset (resp. risky asset) you invest.

The liquidation value (without transaction costs) is the portfolio process

$$V_t = V_t^{\hat{\theta}} = \theta_t^0 S_t^0 + \theta_t S_t, \quad t \in [0, T].$$

In discrete-time $t_0 = 0 < t_1 < \cdots < t_n = T$ a self-financing portfolio process $V = V^{\hat{\theta}}$ satisfies by definition

$$\Delta V_{t_i} = \theta_{t_{i-1}} \Delta S_{t_i}, \quad i = 1, \cdots, n,$$

with the general notation $\Delta X_{t_i} = X_{t_i} - X_{t_{i-1}}$. We then have

$$V_{t_i} = V_0 + \sum_{j=1}^{i} \theta_{t_{i-1}} \Delta S_{t_i}, \quad i = 1, \cdots, n,$$

which is a discrete stochastic integral.

In continuous-time $t \in [0, T]$, a self-financing portfolio process $V = V^{\hat{\theta}}$ satisfies by definition

$$dV_t = \theta_t dS_t, (dV_t \simeq V_{t+dt} - V_t, dt \to 0),$$

which means that

$$V_t = V_0 + \int_0^t \theta_u dS_u,$$

where the stochastic integral needs to be understood for a "nice" process S, i.e. a semi-martingale that allows to give a sense to $\int_0^t \theta_u dS_u$, as a limit of discrete-time stochastic integrals, see discrete-time model.

Let ξ_T be a payoff (terminal wealth) which is promised to be delivered by the seller of a contract.

A hedging (resp. super-hedging) price is the initial value of a self-financing portfolio process $V = V^{\hat{\theta}}$ such that $V_T = \xi_T$ (resp. $V_T \geq \xi_T$) a.s.

Example : Call European option $\xi_T = (S_T - K)^+$, $K > 0$.

Then S_0 is a super-hedging price because, with S_0 buy one unit of the risky asset S and keep it until time T. At time T, you sell it and you get $S_T \geq (S_T - K)^+$.

The goal is to compute the minimal super-hedging price if existence holds or, if possible, a hedging price. Better is to characterize the associated strategy $\hat{\theta}$.

In discrete time $t_0 = 0 < t_1 < \cdots < t_n = T$, suppose that the price process S is a martingale, i.e. $E(S_{t_i}|\mathcal{F}_{t_{i-1}}) = S_{t_{i-1}}$, $i = 1, \cdots, n$. Then, any self-financing portfolio process $V = V^{\hat{\theta}}$ such that $V_T \in L^1$ is a martingale.

In continuous-time, if S is local martingale, then $V = V^{\hat{\theta}}$ is a local-martingale and finally a martingale if $V_T \in L^2$ for Brownian filtrations.

Therefore, if S is a martingale, then the only portfolio $V = V^{\hat{\theta}}$ that could exactly replicate the payoff ξ_T is given by $V_t = E(\xi_T|\mathcal{F}_t)$ so that the hedging price is $V_0 = E(\xi_T)$.

Otherwise, the minimal super-hedging price is $V_0 = \sup_{Q \in \mathcal{M}} E_Q(\xi_T)$, $\mathcal{M}$ the set of all equivalent probability measures under which S is a martingale.

The dynamics of S is $dS_t = \sigma S_t dW_t$, W is a Brownian motion so that S is a martingale.

We may prove that the model is complete, i.e. any payoff $\xi_T \in L^2$ is replicable and there exists just P under which S is a martingale. Therefore, the (unique) replicating price is $V_0 = E(\xi_T)$.

Example : European Call option $\xi_T = (S_T - K)^+$, $K > 0$.

We may show that $V_t = g(t, S_t)$, $t \in [0, T]$, $\theta_t = g_x(t, S_t)$, where g is the solution to the heat equation

$$g_t(t,x) + \frac{\sigma^2 x^2}{2} g_{xx}(t,x) = 0, \quad t \in [0,T), \quad g(T,x) = (x-K)^+.$$

The definition for the dynamics of a self-financing portfolio process needs to be adapted because some transactions costs are charged.

Even if the asset price process is a martingale, a self-financing portfolio process is no more a martingale.

The theory based on the martingale principle seen before is no more valid.

The Leland conjecture for the B&S model with transaction costs

In 1985, **Leland** published the paper

- **Option pricing and replication with transactions costs**, J. Finance, 40 :5 (1985), 1283-1301.

For the Black and Scholes model with proportional transactions costs, he proposed a discrete-time strategy that allows to asymptotically replicate the Call option as the number of revision dates n tends to $+\infty$.

Following the Leland strategy, Leland's conjecture of asymptotic hedging appears to be correct provided that the transaction cost rate $k = k_n$ depends on n such that $k_n \to 0$ rapidly as $n \to \infty$.

The Leland Strategy

Assume :

- The transaction cost rate is $k_n = k_0 n^{-\alpha}$, where $\alpha \geq 0$.
- The price process follows the geometric Brownian motion of the B&S model.

A discrete-time self-financing portfolio process $V = (V_{t_i^n})_{i=0}^n$, revised at each date $0 < t_1^n < \cdots < t_n^n = T$, satisfies :

$$\Delta V_{t_i^n} = \theta_{t_{i-1}^n} \Delta S_{t_i^n} - k_n S_{t_i^n} \left| \theta_{t_i^n} - \theta_{t_{i-1}^n} \right|,$$

The **Leland strategy** for the Call option is defined as :

$$V_0^n = g(0, S_0, \sigma_n), \quad \theta_{t_{i-1}^n} = g_x(t_{i-1}^n, S_{t_{i-1}^n}, \sigma_n),$$

where

$$\sigma_n^2 = \sigma^2 (1 + \frac{\gamma_n}{\sigma}), \quad \gamma_n = \frac{8}{\pi} k_0 n^{\frac{1}{2} - \alpha}.$$

Theorem (Leland-Lott ($\alpha = \frac{1}{2}$) and Kabanov-Safarian ($\alpha \in (0, \frac{1}{2})$), 1997)

Suppose that the transaction cost rate is $k_n = k_0 n^{-\alpha}$ where $\alpha \in (0, \frac{1}{2}]$. Let V^n be the discrete-time self-financing portfolio process generated by the Leland strategy $V_0^n = g(0, S_0, \sigma_n)$, $\theta_{t_{i-1}^n} = g_x(t_{i-1}^n, S_{t_{i-1}^n}, \sigma_n)$. Then,

$$V_T^n \xrightarrow{\mathbb{P}} (S_T - K)^+ \quad \text{as} \quad n \to \infty.$$

☞ This first result has been generalized to other payoff functions, to local and stochastic volatility models. Also, rate of convergence for the mean square error and limit theorem have been obtained (Kabanov, Pergamenchtchikov, Lepinette, Thai N., etc).

Beyond the Leland strategy

- **Setting :**
  - Discrete-time financial market model $(\Omega, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$.
  - One risky asset $S = (S_t)_{t=0}^T$, w.l.o.g the risk-free rate is $r = 0$.

A **self-financing portfolio process** $(V_t)_{t=0}^T$ satisfies by definition the dynamics :

$$\Delta V_t = \theta_{t-1}\Delta S_t - \underbrace{k_{t-1}|\Delta\theta_{t-1}|S_{t-1}}_{\text{⚠Transaction cost term}} \tag{1}$$

- **Motivation :**
  - Presence of proportional transactions costs.
  - No assumption on the price process dynamic.
  - Arbitrary number of revision dates.

Goal

- Find possible portfolios $V = (V_t)_{t=0}^{T} = P(\theta)$ and associated strategies $\theta = (\theta_t)_{t=0}^{T-1}$ such that $V_T \geq g(S_T)$ for some **convex payoff function g.**
- Deduce the infimum super-hedging price

$$P_0^* = \text{ess inf}\{V_0(\theta) : \ \theta = (\theta_t)_{t=0}^{T-1}\}.$$

- **Approach**
  - Essential supremum (resp. infimum).
  - Fenchel conjugate and biconjugate.
  - Conditional supports of the relative prices.

⚠ No use of martingale measure.

Main steps : Step 1

- The super-hedging problem is first solved between time $T-1$ and T for an European claim $\xi_T = g_T(S_T)$ where $g_T \geq 0$ is a convex function without any no-arbitrage condition.

☞ A weak no-arbitrage condition (similar to AIP).

☞ Under the assumption that the conditional supports of the relative prices $\frac{S_{t+1}}{S_t}$ are deterministic intervals, we show that at time $T-1$,

  - There exists minimal super-hedging prices

$$P^*_{T-1} = P^*_{T-1}(\theta_{T-2}, S_{T-1})$$

⚠ **Depends** on the strategy θ_{T-2} chosen at time $T-2$.

We get a specified form $g_{T-1}(\theta_{T-2}, S_{T-1}) = P^*_{T-1}(\theta_{T-2}, S_{T-1})$.

☆ Note that g_{T-1} is still a convex function in S_{T-1}.

Main steps : Step 2

The dependence of g_t, $t \leq T-1$, w.r.t. θ_{t-1} is iteratively identified so that the second step consists in solving the **unusual** super-hedging price problem of the form

$$V_{t-1} + \theta_{t-1}\Delta S_t - k_{t-1}|\Delta\theta_{t-1}|S_{t-1} \geq g_t(\theta_{t-1}, S_t). \tag{2}$$

Our assumption is the following :

- The payoff function g_t at time t is of the form :

$$g_t(\theta_{t-1}, x) = \max_{i=1,\ldots,N} g_t^i(\theta_{t-1}, x) \qquad (\star)$$

  - The mapping $x \mapsto g_t^i(\theta_{t-1}, x)$ is **convex.**
  - $g_t^i(\theta_{t-1}, x) = \hat{g}_t^i(x) - \hat{\mu}_t^i \theta_{t-1} x$

    $(\hat{\mu}_t^i)_{i=1}^N$ are deterministic such that $1 + \hat{\mu}_t^i > 0$.

Main steps : Step 3

Once solved the general problem of Step 2, we need to verify the propagation property that guarantees that the infimum super-hedging price is still a payoff function of the underlying asset in the specified form we have conjectured.

Once again, the infimum super-hedging price could be $-\infty$ for non negative payoff functions ! **The minimal no-arbitrage condition is required at each step, which is a condition between the conditional support of S and the transaction cost rate.**

Main idea !

$$V_{t-1} \geq g_t(\theta_{t-1}, S_t) - \theta_{t-1}\Delta S_t + k_{t-1}|\theta_{t-1} - \theta_{t-2}|S_{t-1},$$
$$\geq \underbrace{\text{ess sup}_{\mathcal{F}_{t-1}}\left(g_t(\theta_{t-1}, S_t) - \theta_{t-1}S_t\right)} + \theta_{t-1}S_{t-1} + k_{t-1}|\Delta\theta_{t-1}|S_{t-1}.$$

$$\text{ess sup}_{\mathcal{F}_{t-1}}\left(g_t(\theta_{t-1}, S_t) - \theta_{t-1}S_t\right) = \max_{i=1,\dots,N} \underbrace{\text{ess sup}_{\mathcal{F}_{t-1}}\left(g_t^i(\theta_{t-1}, S_t) - \theta_{t-1}S_t\right)}$$

$$\begin{aligned}
\text{ess sup}_{\mathcal{F}_{t-1}}\left(g_t^i(\theta_{t-1}, S_t) - \theta_{t-1}S_t\right) &= \sup_{z \in \mathrm{supp}_{\mathcal{F}_{t-1}}(S_t)} \left(\hat{g}_t^i(z) - \hat{\mu}_t^i\theta_{t-1}z - \theta_{t-1}z\right) \\
&= \sup_{x \in K_{t-1}^i} \left(\bar{g}_t^i(x) - \theta_{t-1}x\right) \\
&= \sup_{x \in \mathbb{R}} \left(-\theta_{t-1}x - \bar{f}_t^i(x)\right) \\
&= (\bar{f}_t^i)^*(-\theta_{t-1}).
\end{aligned}$$

Results

- V_{t-1} is a super-hedging price if and only if

$$V_{t-1} \geq \underbrace{f_t^*(-\theta_{t-1}) - \theta_{t-1}S_t + \theta_{t-1}S_{t-1} + k_{t-1}|\Delta\theta_{t-1}|S_{t-1}}_{V_{t-1}(\theta_{t-1})}$$

- The set of all super-hedging prices at time $t-1$ is given by

$$\mathcal{V}_{t-1}(g_t) = \{V_{t-1}(\theta_{t-1}) : \theta_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_{t-1})\} + L^0(\mathbb{R}_+, \mathcal{F}_{t-1})$$

- The infimum super-hedging price is defined as :

$$p_{t-1}(g_t) = \text{ess inf}\, \mathcal{V}_{t-1}(g_t)$$

Main theorem (A. Omrani and E. Lépinette, 2024)

Theorem (A. Omrani and E. Lépinette (2024))

Let g_t be a convex payoff function of the form $(\star)$. Then, the infimum super-hedging price at time $t-1$ satisfies the following :

$$\begin{aligned} p_{t-1}(g_t) &= -(f_{t-1}^* \circ \Phi_{\theta_{t-2}}^{-1})^*(S_{t-1}) \\ &= g_{t-1}(\theta_{t-2}, S_{t-1}), \end{aligned}$$

where $g_{t-1}(\theta_{t-2}, x)$ is a function of the form $(\star)$ and can be deduced backwards from the function g_t.

⚠Distorted Legendre-Fenchel biconjugate makes computations more complicated.

At any time $t-1$, the minimal super-hedging price is $p_{t-1}(g_t) = g_{t-1}(\theta_{t-2}, S_{t-1})$.

Therefore, at time 0, the price is $p_0(g_1) = g_0(\theta_{-1}, S_0)$ where $\theta_{-1} = 0$ as the price is expressed in cash at time $t = 0$. We explicit g_0 from g_1, g_1 from g_2 and so on, since g_T is known as the payoff function.

Given g_0, we deduce θ_0 explicitly so that we can compute $p_1(g_2) = g_1(\theta_0, S_1)$. Precisely, we repeat the procedure : we compute $p_1(g_2) = g_1(\theta_0, S_1)$ from g_2, g_3,..and finally deduce θ_1, etc.

At the end, we get that

$$V_T = V_0 + \sum_{t=1}^{T} \theta_{t-1}\Delta S_t - \sum_{t=1}^{T} k_{t-1}|\Delta\theta_{t-1}|S_{t-1} \geq g(T, S_T) = g(S_T).$$

Let Γ_t be a family of $\mathcal{F}_t$-measurable random variables with values in $\mathbf{R} \cup \{+\infty\}$. There exists a unique $\mathcal{F}_{t-1}$-measurable random variable denoted by $\text{ess sup}_{\mathcal{F}_{t-1}} \Gamma_t$ that satisfies the following statements :

1) $\text{ess sup}_{\mathcal{F}_{t-1}} \Gamma_t \geq \gamma_t$ a.s. for any $\gamma_t \in \Gamma_t$,
2) If γ_{t-1} is a $\mathcal{F}_{t-1}$-measurable random variable such that $\gamma_{t-1} \geq \gamma_t$ a.s. for any $\gamma_t \in \Gamma_t$, then $\gamma_{t-1} \geq \text{ess sup}_{\mathcal{F}_{t-1}} \Gamma_t$.

The main mathematical tools in this new approach

Conditional support

Let S_t be a $\mathcal{F}_t$-measurable random variable. There exists a smallest $\mathcal{F}_{t-1}$-measurable random set denoted by $\mathrm{supp}_{\mathcal{F}_{t-1}} S_t$ such that

$$P(S_t \in \mathrm{supp}_{\mathcal{F}_{t-1}} S_t) = 1,$$

$$Graph(\mathrm{supp}_{\mathcal{F}_{t-1}} S_t) = \{(\omega, x) \in \Omega \times \mathbf{R} : \; x \in \mathrm{supp}_{\mathcal{F}_{t-1}} S_t(\omega)\} \in \mathcal{F}_{t-1} \times \mathcal{B}(\mathbf{R}).$$

Let $h_{t-1}(\omega, x)$ be a random function such that, a.s(ω), $h_{t-1}(\omega, \cdot)$ is l.s.c. and for, every x, $h_{t-1}(\cdot, x)$ is $\mathcal{F}_{t-1}$ measurable. Then, for any $\mathcal{F}_t$ measurable random variable S_t, we have

$$\text{ess sup}_{\mathcal{F}_{t-1}} h_{t-1}(\omega, S_t(\omega)) = \sup_{x \in \text{supp}_{\mathcal{F}_{t-1}} S_t(\omega)} h(\omega, x).$$

The proof (by Lepinette and co-authors) uses some arguments from the theory of random set, see the book of Molchanov.

The end...

Thank you for your attention ! [1]

1. E. Lepinette, A. Omrani. Beyond the Leland strategies. Submitted. Hal-04974143v1.