On Schur-like properties

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6. Introduce the notion of a weak p-convergent operator on Banach spaces and lattices.
7. Introduce the notion of a disjoint p-convergent operator on Banach lattices.
8. Apply these operators to a study of the positive Schur property of order p.

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2. These are "ℓ_p-versions" of the well-known Schur and positive Schur properties on Banach spaces and Banach lattices, respectively.
3. We discuss examples of spaces which have or do not have these properties and give a condition on a Banach lattice such that these two properties coincide.
4. We introduce and study the notions of a disjoint p-convergent operator on Banach lattices, as well as a weak p-convergent operator on Banach spaces (with its positive variant in Banach lattices) and consider their applications to a study of the positive Schur property of order p.

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Preliminaries

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- As is custom, we agree to use E, F, G etc. to denote Banach lattices.
- In this talk we will throughout assume that the Banach lattices are real, i.e. they are linear spaces over $\mathbb{R}$.

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- Let $1 \leq p < \infty$. The conjugate number will be denoted by p', i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

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- The unit coordinate vector e_n in these sequence spaces, is the sequence $e_n = (\delta_{n,j})_j$, where $\delta_{n,j} = 0$ if $j \neq n$ and $\delta_{n,n} = 1$.

Definition

Let $1 \leq p < \infty$. A sequence (x_n) in X is said to be weakly p-summable (or it is said to be a weak ℓ_p sequence) if $(\langle x^*, x_n \rangle) \in \ell_p$ for each $x^* \in X^*$.

Remark

- We denote the vector space of all weakly p-summable sequences in X by $\ell_p^{weak}(X)$.

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- Since X is a Banach space, then so is $\ell_p^{weak}(X)$, with norm

$$\|(x_n)\|_p^{weak} = \sup\{(\sum_{n=1}^{\infty} |\langle x^*, x_n\rangle|^p)^{\frac{1}{p}} : x^* \in \mathcal{B}_{X^*}\}.$$

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- For a given $(x_n) \in \ell_p^{weak}(X)$, the mapping

$$R_{(x_i)} : \ell_{p'} \to X : (\xi_i) \mapsto \sum_i \xi_i x_i$$

is bounded and linear and $(x_i) \mapsto R_{(x_i)}$ identifies $\ell_p^{weak}(X)$ and $\mathcal{L}(\ell_{p'}, X)$ isometrically.

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- Similarly, $\ell_1^{weak}(X) = \mathcal{L}(c_0, X)$ isometrically.

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- It is a Banach space with respect to the norm

$$\|(x_n)\|_{c_0^{weak}} := \sup_{\|x^*\|\leq 1} \|(\langle x^*, x_n\rangle)\|_{c_0}$$

and may be isometrically identified with $\mathcal{W}(\ell_1, X)$.

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- The closure in $\ell_p^{weak}(X)$ of the set of all sequences in X which have only a finite number of non-zero terms, is a Banach space with respect to the norm $\|\cdot\|_p^{weak}$.
- We denote this space by $\ell_p^u(X)$ and recall that $(x_n) \mapsto R_{(x_n)}$ identifies the spaces $\ell_p^u(X)$ and $\mathcal{K}(\ell_{p'}, X)$ isometrically for all $1 < p < \infty$ and it identifies $\ell_1^u(X)$ isometrically with $\mathcal{K}(c_0, X)$

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- The elements of $\ell_p^u(X)$ are called the unconditionally p-summable sequences in X.
- It is well-known that $(x_n) \in \ell_1^u(X)$ if and only if (x_n) is an unconditionally summable sequence in X.

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- The elements of $\ell_p^u(X)$ are called the unconditionally p-summable sequences in X.
- It is well-known that $(x_n) \in \ell_1^u(X)$ if and only if (x_n) is an unconditionally summable sequence in X.
- Then $\ell_1^{weak}(X) = \ell_1^u(X)$ if and only if X does not contain a copy of c_0.

Definition

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- A bounded subset $A \subset X$ is said to be relatively weakly p-compact if every sequence in A has a weakly p-convergent subsequence.

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- This is equivalent to say that T is p-convergent if it maps relatively weakly p-compact sets of X into relatively norm compact sets of Y.

Definition

Recall from Castillo and Sanchez, that a Banach space X is said to have the Dunford-Pettis property of order p (DPP_p for short) if for $(x_n) \in \ell_p^{weak}(X)$ and $(x_n^*) \in c_0^{weak}(X^*)$ we have $\langle x_n^*, x_n \rangle \to 0$ as $n \to \infty$.

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Weakly p-summable sequences in Banach lattices

Remark

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- Given $x^* \in E^*$, then we have $(\langle (x^*)^+, x_n \rangle) \in \ell_p$ and $(\langle (x^*)^-, x_n \rangle) \in \ell_p$.
- Also $\langle (x^*)^+, y_n \rangle \leq \langle (x^*)^+, x_n \rangle$ and $\langle (x^*)^-, y_n \rangle \leq \langle (x^*)^-, x_n \rangle$ for all n.

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- Thus $(\langle x^*, y_n \rangle) \in \ell_p$.
- This shows that $(y_n) \in \ell_p^{weak}(E)$ as well.

Some facts

- It is not always true that if $x_n \xrightarrow[\infty]{n} 0$ weakly, then $|x_n| \xrightarrow[\infty]{n} 0$ weakly, that is, in general, the lattice operations are not necessarily weakly sequentially continuous.

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Some facts

- It is not always true that if $x_n \xrightarrow[\infty]{n} 0$ weakly, then $|x_n| \xrightarrow[\infty]{n} 0$ weakly, that is, in general, the lattice operations are not necessarily weakly sequentially continuous.
- It is shown that if the elements of a sequence (x_i) in a Banach lattice E are pairwise disjoint, then $(x_i) \in \ell_p^{weak}(E) \iff (|x_i|) \in \ell_p^{weak}(E)$.

Remark

- Let (x_i) be a sequence in a Banach lattice E, such that for all $(\lambda_i) \in \ell_p$ the corresponding series $\sum_{i=1}^{\infty} \lambda_i x_i$ converges in E.

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- Then, since for each $(\lambda_i) \in \ell_p$, also $(|\lambda_i|) \in \ell_p$, it follows that $(\lambda_i^+), (\lambda_i^-) \in \ell_p$.
- Thus, the series $\sum_{i=1}^{\infty} |\lambda_i| x_i$, $\sum_{i=1}^{\infty} \lambda_i^+ x_i$ and $\sum_{i=1}^{\infty} \lambda_i^- x_i$ converge in E as well.

Proposition

Let E be a Banach lattice and suppose that (x_i) is a sequence in E such that the elements x_i are pairwise disjoint. Then we have

$$(x_i) \in \ell_p^{weak}(E) \iff (|x_i|) \in \ell_p^{weak}(E).$$

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Corollary

Let E be a Banach lattice and suppose that the elements x_i of a sequence $(x_i) \subset E$ are pairwise disjoint. Then we have

$$(x_i) \in \ell_p^{weak}(E) \iff (x_i^+), (x_i^-) \in \ell_p^{weak}(E).$$

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Remark

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- Therefore, Zeekoei and Fourie introduced the following concept:

Definition

We say a Banach lattice E is weak p-consistent (for $1 \leq p < \infty$) if it follows from $(x_i) \in \ell_p^{weak}(E)$ that $(|x_i|) \in \ell_p^{weak}(E)$.

Remark

Since we shall need the lattice operations to satisfy a seemingly weaker property than being weakly sequentially continuous later on, we introduce the notion "weakly sequentially p-continuous" as follows:

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Definition

The lattice operations in a Banach lattice E are said to be weakly sequentially p-continuous if the sequence $(|x_n|)$ converges weakly to 0 for every weakly p-summable sequence (x_n).

The Schur and Positive Schur properties of order p

Definition

A Banach lattice E is said to have the positive Schur property of order p (briefly, E has the SP_p^+) if each sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, is norm convergent to 0.

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Remark

If we agree to say that E has the SP_∞^+ if each sequence $(x_n) \in c_0^{weak}(E)$ with positive terms, is norm convergent to 0, then we may assume $1 \leq p \leq \infty$ in the definition, the SP_∞^+, however, will then coincide with the well known positive Schur property.

Lemma

If in a Banach lattice E there exists a sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, which is not norm convergent to 0, then there exists a sequence $(z_k) \in \ell_p^{weak}(E)$ such that $z_n \geq 0$ for all n, $z_n \wedge z_m = 0$ for all $m \neq n$ and $\|z_n\| \nrightarrow 0$.

Proof

- Suppose that in a Banach lattice E there exists a sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, which is not norm convergent to 0.

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- Thus, we may assume that $c := \inf_n \|x_n\| > 0$.

Proof

- Suppose that in a Banach lattice E there exists a sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, which is not norm convergent to 0.
- Thus, we may assume that $c := \inf_n \|x_n\| > 0$.
- Then, putting $y_n = c^{-1}x_n$ for all n we find a subsequence (y_{nk}), a constant $d > 0$, and a sequence (z_k) of pairwise disjoint elements such that $0 < z_k \leq y_{n_k}$ and $\|z_k\| \geq d$ for all k.

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Proof

- Suppose that in a Banach lattice E there exists a sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, which is not norm convergent to 0.
- Thus, we may assume that $c := \inf_n \|x_n\| > 0$.
- Then, putting $y_n = c^{-1}x_n$ for all n we find a subsequence (y_{nk}), a constant $d > 0$, and a sequence (z_k) of pairwise disjoint elements such that $0 < z_k \leq y_{n_k}$ and $\|z_k\| \geq d$ for all k.
- The sequence (y_n) belongs to $\ell_p^{weak}(E)$ and therefore, also $(z_k) \in \ell_p^{weak}(E)$.

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It therefore follows that:

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Proposition

A Banach lattice E has the positive Schur property of order p if and only if each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, is norm convergent to 0.

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Proposition

A Banach lattice E has the positive Schur property of order p if and only if each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, is norm convergent to 0.

Definition

A Banach space X is said to have the Schur property of order p (briefly, X has the SP_p) if every weakly p-summable sequence is norm convergent to 0.

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Remark

1. It follows from the literature that every weakly p-summable sequence in a Banach space X is norm convergent to 0 (for $1 \leq p < \infty$) if and only if $\ell_p^{weak}(X) = \ell_p^u(X)$.

Remark

1. It follows from the literature that every weakly p-summable sequence in a Banach space X is norm convergent to 0 (for $1 \leq p < \infty$) if and only if $\ell_p^{weak}(X) = \ell_p^u(X)$.
2. As was mentioned in the preliminaries, it is a well-known fact that $\ell_1^{weak}(X) = \ell_1^u(X)$ if and only if X contains no copy of c_0.

We conclude that:

Proposition

Let X be a Banach space which contains no copy of c_0. Then X has the SP_1.

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Corollary

In each Banach lattice E which contains no copy of c_0, the lattice operations are weakly sequentially 1-continuous.

Remark

In general, the weak sequential continuity of the lattice operations in a Banach lattice is not implied by the weakly sequentially p-continuity of the same, as is illustrated by the following result:

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Proposition

The space $L_1[0,1]$ has SP_1. Thus, the lattice operations in $L_1[0,1]$ are weakly sequentially 1-continuous, but they are not weakly sequentially continuous.

Examples

- Let $1 \leq p < \infty$.
  Since every weak ℓ_1-sequence in an L_p-space is a strong ℓ_r sequence where $r = \max\{p, 2\}$, it follows that any L_p-space has SP_1.
  Thus, the lattice operations in $L_p[0, 1]$ are weakly sequentially 1-continuous, but they are not weakly sequentially continuous.

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  Since every weak ℓ_1-sequence in an L_p-space is a strong ℓ_r sequence where $r = \max\{p, 2\}$, it follows that any L_p-space has SP_1.
  Thus, the lattice operations in $L_p[0, 1]$ are weakly sequentially 1-continuous, but they are not weakly sequentially continuous.
- In any Banach space X which has cotype q (where $2 \leq q < \infty$) every weak ℓ_1 sequence is a strong ℓ_q sequence.
  Thus, all Banach spaces with finite cotype, have SP_1.

Remark

- Following our discussions, it is clear that a Banach space X has the SP_p (for $1 < p < \infty$) if and only if each bounded linear operator from $\ell^{p'}$ to X is compact.

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- Following our discussions, it is clear that a Banach space X has the SP_p (for $1 < p < \infty$) if and only if each bounded linear operator from $\ell^{p'}$ to X is compact.
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Remark

- Following our discussions, it is clear that a Banach space X has the SP_p (for $1 < p < \infty$) if and only if each bounded linear operator from $\ell^{p'}$ to X is compact.
- Similarly, a Banach space X has the SP_1 if each bounded linear operator from c_0 to X is compact.
- These observations provide us with an abundance of examples of Banach spaces which have the SP_p for some $1 \leq p < \infty$, but which do not have the Schur property:

Examples

- Suppose $1 < p < \infty$ and $1 < q < p'$ (i.e. $\frac{1}{q} + \frac{1}{p} > 1$), then by Pitt's Theorem each bounded linear operator $T : \ell_{p'} \to \ell_q$ is compact. Thus, all the spaces ℓ_q have SP_p. In fact, more is true:
  Using a version of Pitt's Theorem, it follows that all closed subspaces of ℓ_q (for all $q < p'$) have SP_p.
  Of course, none of the spaces ℓ_q have the Schur property.

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  Using a version of Pitt's Theorem, it follows that all closed subspaces of ℓ_q (for all $q < p'$) have SP_p.
  Of course, none of the spaces ℓ_q have the Schur property.
- Since it is also well-known that all bounded linear operators from c_0 into ℓ_p (for all $1 \leq p < \infty$) are compact (or since ℓ_p does not contain a copy of c_0), it follows that all the ℓ_p-spaces have the SP_1.
  Again, none of the ℓ_p-spaces (for $p > 1$) have the Schur property.

We also have the following example of a space with the DPP which does not have SP_2:

Example

- Let (Ω, Σ, μ) be some probability space.

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- By the above discussion, every weakly 2-summable sequence in $L_1(\mu)$ would be a norm null sequence if and only if each bounded linear operator from the sequence space ℓ_2 to $L_1(\mu)$ were compact.

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- This is impossible, since for instance we know that ℓ_2 embeds isometrically in $L_1(\mu)$.
- Thus, there has to be a weakly 2-summable sequence which is not norm null, showing that $L_1(\mu)$ does not have the SP_2.

More examples of L_p-spaces without the Schur property of order p for some choices of p:

Examples

- For $1 \leq r \leq 2$, $\ell_{p'}$ embeds in L_r if and only if $r \leq p' \leq 2$. Thus, L_r does not have SP_p for all $2 \leq p \leq r'$.

More examples of L_p-spaces without the Schur property of order p for some choices of p:

Examples

- For $1 \leq r \leq 2$, $\ell_{p'}$ embeds in L_r if and only if $r \leq p' \leq 2$. Thus, L_r does not have SP_p for all $2 \leq p \leq r'$.
- For $2 < r < \infty$, $\ell_{p'}$ embeds in L_r if and only if $p' = 2$ or $p' = r$. Thus, L_r does not have SP_p for $p = 2$ or $p = r'$.

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(ii) E has the SP_p^+.

Proof

- It is clear that if a Banach lattice E has the SP_p, then the lattice operations are weakly sequentially p-continuous and E has the SP_p^+.

Proof

- It is clear that if a Banach lattice E has the SP_p, then the lattice operations are weakly sequentially p-continuous and E has the SP_p^+.
- On the other hand, if E is a weak p-consistent Banach lattice and E has the SP_p^+, then for each $(x_n) \in \ell_p^{weak}(E)$ we have $(|x_n|) \in \ell_p^{weak}(E)$ and so $\|x_n\| = \||x_n|\| \to 0$ as $n \to \infty$.

weak p-convergent operators, disjoint p-convergent operators and the SP_p^+

Remark

Throughout this section we assume that $1 \leq p < \infty$, unless otherwise stated.

Definition

Let E, F be Banach lattices. An operator $T : E \to F$ is said to be *disjoint p-convergent* if $\|Tx_n\| \to 0$ for all weakly p-summable sequences (x_n) so that the elements x_i are pairwise disjoint.

Proposition

Let E, F be Banach lattices. If an operator $T : E \to F$ satisfies $\|Tx_n\| \to 0$ for all sequences $(x_n) \in \ell_p^{weak}(E)$ such that $x_n \wedge x_m = 0$ if $m \neq n$, then T is disjoint p-convergent.

Proof

- Suppose an operator $T : E \to F$ satisfies $\|Tx_n\| \to 0$ for all sequences $(x_n) \in \ell_p^{weak}(E)$ such that $x_n \wedge x_m = 0$ if $m \neq n$.

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- Suppose an operator $T : E \to F$ satisfies $\|Tx_n\| \to 0$ for all sequences $(x_n) \in \ell_p^{weak}(E)$ such that $x_n \wedge x_m = 0$ if $m \neq n$.
- Now assume that $(y_n) \in \ell_p^{weak}(E)$ and the elements y_i are pairwise disjoint, i.e. $|y_m| \wedge |y_n| = 0$ for $m \neq n$.

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- Suppose an operator $T : E \to F$ satisfies $\|Tx_n\| \to 0$ for all sequences $(x_n) \in \ell_p^{weak}(E)$ such that $x_n \wedge x_m = 0$ if $m \neq n$.
- Now assume that $(y_n) \in \ell_p^{weak}(E)$ and the elements y_i are pairwise disjoint, i.e. $|y_m| \wedge |y_n| = 0$ for $m \neq n$.
- Then $y_n^+ \wedge y_m^+ = 0$ and $y_n^- \wedge y_m^- = 0$ for $m \neq n$ and we have $(y_n^+) \in \ell_p^{weak}(E)$ and $(y_n^-) \in \ell_p^{weak}(E)$.

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Proof

- Suppose an operator $T : E \to F$ satisfies $\|Tx_n\| \to 0$ for all sequences $(x_n) \in \ell_p^{weak}(E)$ such that $x_n \wedge x_m = 0$ if $m \neq n$.
- Now assume that $(y_n) \in \ell_p^{weak}(E)$ and the elements y_i are pairwise disjoint, i.e. $|y_m| \wedge |y_n| = 0$ for $m \neq n$.
- Then $y_n^+ \wedge y_m^+ = 0$ and $y_n^- \wedge y_m^- = 0$ for $m \neq n$ and we have $(y_n^+) \in \ell_p^{weak}(E)$ and $(y_n^-) \in \ell_p^{weak}(E)$.
- Therefore, both $\|Ty_n^+\| \to 0$ and $\|Ty_n^-\| \to 0$ as $n \to \infty$, and it follows that

$$\|Ty_n\| = \|Ty_n^+ - Ty_n^-\| \leq \|Ty_n^+\| + \|Ty_n^-\| \to 0 \text{ if } n \to \infty.$$

□

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Moreover, the above argument also shows that:

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Proposition

Let E, F be Banach lattices. An operator $T : E \to F$ is disjoint p-convergent if and only if $\|Tx_n\| \to 0$ for all sequences $(x_n) \in \ell_p^{weak}(E)$ consisting of pairwise disjoint positive elements.

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Remark

It follows that a Banach lattice E has the positive Schur property of order p if and only if the id_E is disjoint p-convergent.

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An operator $T : X \to Y$ is said to be weak p-convergent if $y_n^*(Tx_n) \xrightarrow[\infty]{n} 0$ for every $(x_n) \in \ell_p^{weak}(X)$ and every $(y_n^*) \in c_0^{weak}(Y^*)$.

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Remark

If X has the DPP_p, then id_X is weak p-convergent.

Remarks

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- However, since $(e_n) \in \ell_p^{weak}(c_0)$, (e_n) is a disjoint sequence in c_0 and $\|e_n\| \nrightarrow 0$, it follows that id_{c_0} is not disjoint p-convergent.

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- Suppose a Banach space Y has DPP_p. Let X be any Banach space and $T : X \to Y$ a bounded linear operator.

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- However, since $(e_n) \in \ell_p^{weak}(c_0)$, (e_n) is a disjoint sequence in c_0 and $\|e_n\| \nrightarrow 0$, it follows that id_{c_0} is not disjoint p-convergent.
- Suppose a Banach space Y has DPP_p. Let X be any Banach space and $T : X \to Y$ a bounded linear operator.
- Consider $(x_n) \in \ell_p^{weak}(X)$. Then $(Tx_n) \in \ell_p^{weak}(Y)$. Therefore, if $(y_n^*) \subset Y^*$, $y_n^* \to 0$ weakly, then $y_n^*(Tx_n) \xrightarrow[\infty]{n} 0$, so T is weak p-convergent.

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The reader is referred to standard textbooks on Banach lattices to recall the definitions of the following well-known notions:

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- Banach lattices with order continuous norm.
- A Kantorovich-Banach space, or briefly a KB-space.
- Dual KB-spaces.
- Dedekind σ-completeness of a vector lattice.

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   (a) E has the SP_p^+.

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Let E and F be two Banach lattices with F Dedekind σ-complete. If each positive weak p-convergent operator $T : E \to F$ is disjoint p-convergent, then one of the following assertions is valid:

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Thank you.

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