A short note on pricing with integer-valued strategies and some vector lattices translations

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Introduction

In financial mathematics, probability and stochastics are just the beginning.

Finance has a way of pulling in analysis and linear algebra, it sneaks in functional analysis, optimization, operator theory, convexity, topology ...

In this field, you never really know enough :'(

But using your background, you can add your own touch!

Outline of the presentation

1. The classical financial market model and the AIP condition
2. The specific case of integer valued strategies
3. The Riesz space generalization

Setting

We consider

- $\mathcal{F}$ a complete σ-algebra: modeling the whole information of the market.
- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space where Ω is the space of all possible states of the market.
- An interval $[0, T] \subset \mathbf{R}$ that represents the period of trading.
- $(\mathcal{F}_t)_{t\in[0,T]}$ a complete and right continuous filtration such that $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_t$ contains the information available at time $t \in [0, T]$.

Financial market model

- d risky assets.
- Prices process: $(S_t = (S_t^1, ..., S_t^d))_{t\in[0,T]} \subset \mathbb{L}^0(\mathbf{R}_+^d, \mathcal{F}_t)$.
- Quantities process: $(\theta_t = (\theta_t^1, ..., \theta_t^d))_{t\in[0,T]} \in \mathbb{L}^0(\mathbf{R}^d, \mathcal{F}_t)$ called strategy.
- The portfolio values at time t: $V_t = S_t\theta_t = \sum_{k=1}^{d} \theta_t^k S_t^k$.
- A self-financing discrete time portfolio process $(V_t)_{0\leq t\leq T}$ satisfies:

$$\Delta V_t = V_t - V_{t-1} = \theta_{t-1}\Delta S_t, \quad t = 1, \cdots, T.$$

And its terminal value at time T, starting at time t, is then

$$V_{t,T} = \sum_{i=t}^{T} \Delta V_i = \sum_{i=t}^{T} \theta_{i-1}\Delta S_i.$$

Elementary portfolio processes

Definition (Cherif-Lepinette)

We denote by $\mathcal{V}_{t,T}$ a set of T terminal elementary portfolios, starting with zero initial capital at time t.
Typical example of model:

$$\mathcal{V}_{t,T}^{det} = \left\{ V_{t,T} = \sum_{i=1}^{n} \theta_{t_{i-1}} \Delta S_{t_i},\ t = t_0 < .... < t_n = T,\ \theta_{t_i} \in \mathbb{L}^0(\mathbf{R}^d, \mathcal{F}_{t_i}) \right\},$$

or more generally:

$$\mathcal{V}_{t,T}^{rand} = \left\{ \sum_{i=1}^{n} \theta_{\tau_{i-1}} \Delta S_{\tau_i},\ t = \tau_0 < .. < \tau_n = T,\ \tau_i \in \mathcal{T}_{t,T},\ \theta_{\tau_i} \in \mathbb{L}^0(\mathbf{R}^d, \mathcal{F}_{\tau_i}) \right\},$$

$\mathcal{T}_{t,T}$ denotes the set of all $[t, T]$-valued stopping times.

The conditional supremum

The **conditional essential supremum** of a random variable $X \in \mathbb{L}^0(\mathbf{R}, \mathcal{F})$ given a sub-σ-algebra $\mathcal{H}$ is the unique $\mathcal{H}$-measurable random variable denoted by $\text{ess sup}_{\mathcal{H}}(X)$ which satisfies

$$\text{ess sup}_{\mathcal{H}}(X) = \inf\{Y \in \mathbb{L}^0(\mathbf{R}, \mathcal{H}) \text{ such that } X \leq Y\},$$

where the infimum is defined w.r.t the order relation: $\leq$ a.s.

We define **the conditional essential infimum** as

$$\text{ess inf}_{\mathcal{H}}(X) = -\,\text{ess sup}_{\mathcal{H}}(-X).$$

Basic properties of the CS

Let

$$\begin{aligned} \text{ess sup}_{\mathcal{H}} : \mathbb{L}^0(\mathbf{R}, \mathcal{F}) &\longrightarrow \mathbb{L}^0(\overline{\mathbf{R}}, \mathcal{H}) \\ X &\longmapsto \text{ess sup}_{\mathcal{H}}(X) \end{aligned}$$

Consider $f, g \in \mathbb{L}^0(\mathbf{R}, \mathcal{F})$ and two sub-σ-algebras $\mathcal{H} \subset \mathcal{H}'$.

1. $\text{ess sup}_{\mathcal{H}}$ is positive, idempotent and increasing.
2. $\text{ess sup}_{\mathcal{H}}(f+g) \leq \text{ess sup}_{\mathcal{H}}(f) + \text{ess sup}_{\mathcal{H}}(g)$.
3. For $g \in \mathbb{L}^0(\mathbf{R}, \mathcal{H})$, $\text{ess sup}_{\mathcal{H}}(f+g) = \text{ess sup}_{\mathcal{H}}(f) + g$.
4. $\text{ess sup}_{\mathcal{H}}(hf) = h\,\text{ess sup}_{\mathcal{H}}(f)$ for any $h \in \mathbb{L}^0(\mathbf{R}^+, \mathcal{H})$.
5. $\text{ess sup}_{\mathcal{H}'}\,\text{ess sup}_{\mathcal{H}}(f) = \text{ess sup}_{\mathcal{H}'}\,\text{ess sup}_{\mathcal{H}}(f) = \text{ess sup}_{\mathcal{H}}(f)$.

Super-hedging problem

Definition

A contingent claim or payoff is a terminal wealth that must be delivered at some maturity date T to the holder of some financial contract.
A payoff $\xi_T \in \mathbb{L}^0(\mathbf{R}, \mathcal{F}_T)$ is said to be super-hedgeable at time t, if there exists an initial capital $p_t \in \mathbb{L}^0(\mathbf{R}, \mathcal{F}_t)$ (called a super-hedging price) and a portfolio process $V_{t,T}$ such that

$$p_t + V_{t,T} \geq \xi_T.$$

We denote by $\mathcal{P}_{t,T}(\xi_T)$ the set of super-hedging prices for a payoff $\xi_T \in \mathbb{L}^0(\mathbf{R}, \mathcal{F}_T)$.

AIP condition [1]

The condition we introduce avoids infinite negative prices.

Definition (Absence of immediate profit AIP)

An immediate profit at time $t < T$ is the possibility to make a profit from a negative initial capital. On the contrary, we say that the Absence of Immediate Profit (AIP) holds, ie:

$$\mathcal{P}_{t,T}(0) \cap \mathbb{L}^0(\mathbf{R}_-, \mathcal{F}_t) = \{0\}, \forall t \leq T.$$

Proposition

$AIP \iff \text{ess}\inf(\mathcal{P}_{t,T}(0)) = 0 \iff \mathcal{P}_{t,T}(0) = \mathbb{L}^0(\mathbf{R}_+, \mathcal{F}_t).$

[1] Baptiste, J., Carassus, L. and Lépinette E., Pricing without martingale measure, 2018. Journal of Mathematical Analysis and Applications

Proposition

The AIP condition holds if and only if , for any $t \leq T$ and for all $V_{t,T} \in \mathcal{V}_{t,T}$,

$$\text{ess inf}_{\mathcal{F}_t}(V_{t,T}) \leqslant 0.$$

Theorem (Cherif-Lepinette)

Suppose that $d = 1$. AIP condition holds if and only if

$$\text{ess inf}_{\mathcal{F}_{t_1}}(S_{t_2}) \leq S_{t_1} \leq \text{ess sup}_{\mathcal{F}_{t_1}}(S_{t_2}), \quad \forall\ t_1 \leq t_2, t_1, t_2 \in \mathcal{T}_{t,T}.$$

$\mathcal{T}_{t,T}$ denotes the set of all $[t, T]$-valued stopping times.

Dynamical Pricing Approach

We denote by $P_{t,t+1}(g_{t+1})$ the set of all one step super-hedging prices at time t for the payoff $g_{t+1} \in L^0(\mathbf{R}^d, F_{t+1})$ and the corresponding one step infimum super-hedging price:

$$p_{t,t+1}(g_{t+1}) := essinf_{F_{t+1}} P_{t,t+1}(g_{t+1}).$$

The Dynamic Programming states , under the assumption that $p_{t,T}$ is actually a price, that

$$p_{t,T}(g_T) = p_{t,t+1}(p_{t+1,T}(g_T))$$

Dynamical Pricing Approach

Suppose the model satisfies the following conditions:

$$\text{ess inf}_{\mathcal{F}_{t-1}} S_t = k_t^d S_{t-1} \quad \text{a.s.}$$
$$\text{ess sup}_{\mathcal{F}_{t-1}} S_t = k_t^u S_{t-1} \quad \text{a.s.}$$

where $(k_t^d)_{t=1}^T$, $(k_t^u)_{t=1}^T$, and S_0 are deterministic, non-negative values.

AIP Condition:.

AIP holds true if and only if: $\forall t \in \{1, \dots, T\}$:

- $k_t^d \in [0, 1]$
- $k_t^u \in [1, +\infty)$

The super-hedging problem with integer-valued strategies:(Cherif-El Mansour-Lepinette)

- We suppose that $\theta_t \in \mathbb{L}^0(\mathbb{Z}, \mathcal{F}_t)$, $\forall t \in \{0, ..., T\}$.
- ess $\sup_{\mathcal{F}_{t-1}}(S_t) = k^u_{t-1} S_{t-1}$, ess $\inf_{\mathcal{F}_{t-1}}(S_t) = k^d_{t-1} S_{t-1}$, $\forall t \in [0, T]$. for some $k^d_{t-1} \in (0, 1)$ and $k^u_{t-1} \in (1, \infty)$.

$\Rightarrow$ Our goal is to compute the set of all V_0, initial values of self-financing portfolio processes $(V_t)_{0 \le t \le T}$, such that $V_T \geq \xi_T$. See [3].

One step problem (Cherif-El Mansour-Lepinette)

We need a portfolio process V_{t-1} and a strategy $\theta_{t-1} \in \mathbb{L}^0(\mathbb{Z}, \mathcal{F}_{t-1})$ such that:

$$V_t = V_{t-1} + \theta_{t-1}\Delta S_t = V_{t-1} + V_{t-1}^t \geq g_t(S_t)$$

$$\Leftrightarrow V_{t-1} \geq \text{ess sup}_{\mathcal{F}_{t-1}}(g_t(S_t) - \theta_{t-1}S_t) + \theta_{t-1}S_{t-1}$$

We denote $\tilde{V}_{t-1}(\theta_{t-1}) = \text{ess sup}_{\mathcal{F}_{t-1}}(g_t(S_t) - \theta_{t-1}S_t) + \theta_{t-1}S_{t-1}$.

We define V^*_{t-1} as the infimum of all the superhedging prices at time $t-1$ over all integer-valued strategies in $\mathbb{Z}$, i.e. We denote

$$V^*_{t-1} := \operatorname*{ess\,inf}_{\theta_{t-1} \in \mathbb{L}^0(\mathbb{Z}, \mathcal{F}_{t-1})} \tilde{V}_{t-1}(\theta_{t-1}).$$

We show that

$$V^*_{t-1} = \inf_{\theta \in \mathbb{Z}} \tilde{V}_{t-1}(\theta).$$

The optimal strategy ie the strategy associated to the infimum super-hedging price is denoted:

$$\theta^*_{t-1} = argmin(V^*_{t-1}).$$

The case of the call option

The European Call option contract corresponds to the possibility to buy the risky asset at price K instead of S_T. Clearly, it is interesting to exercice it if and only if $S_T \geq K$, so the payoff if $g(S_T) = (S_T - K)^+$.

Example in the one step problem: the case of the Call option (Cherif-El Mansour-Lepinette)

Suppose that $\xi_T^n = ng(S_T)$ whith $n \geq 1$, $g(x) = (x-K)^+$ with $K = 500$, $k^d = 0.9$ and $k^u = 1.2$.
Observe that

$$V_{T-1} + \theta_{T-1}\Delta S_T \geq ng(S_T)$$
$$\Updownarrow$$
$$V_{T-1} \geq V_{T-1}(\theta_{T-1}) = \max_{k\in\{k^d,k^u\}}[ng(kS_{T-1}) - \theta_{T-1}kS_{T-1}] + \theta_{T-1}S_{T-1}.$$

In this case we obtain the explicit expression of $V_{T-1}(\theta_{T-1})$ and deduce

$$V^*_{T-1} = \inf_{\theta\in\mathbb{Z}} V_{T-1}(\theta).$$

A graphic illustration of $V^{*,n}_{T-1}/n$ as a function of S_{T-1}: (Cherif-El Mansour-Lepinette)

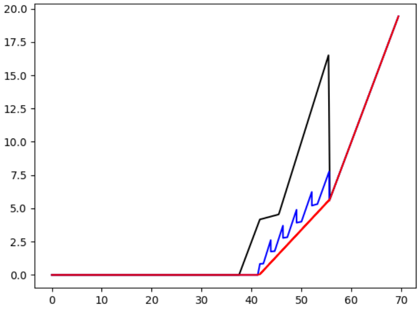

Figure: The function $x = S_{T-1} \mapsto g(x,n)/n = V^{*,n}_{T-1}/n$.
For $n = 1$ (black), $n = 5$ (blue), $n = 100$ (red).

- $V^{*,n}_{T-1}$ is a continuous piecewise affine but not convex function of S_{T-1}

The general result

Theorem (Cherif-El Mansour-Lepinettel)

- *Suppose that, at time $T > 0$, the payoff is $\xi_T = g_T(S_T)$ where g_T is a continuous piecewise affine function.*
- *We assume that there exists deterministic numbers $k^d_{t-1} \in (0,1)$ and $k^u_{t-1} \in (1,\infty)$ for each $t = 1, \cdots, T$ such that we have*

$$\text{ess sup}_{\mathcal{F}_{t-1}}(S_t) = k^u_{t-1} S_{t-1}, \quad \text{ess inf}_{\mathcal{F}_{t-1}}(S_t) = k^u_{t-1} S_{t-1}.$$

$\Rightarrow$ *The minimal super-hedging portfolio process $(V^*_t)_{t=0,\cdots,T}$ exists, such that $V^*_T \geq \xi_T$.*

$\Rightarrow$ $V^*_t = g(t, S_t)$ *where $g(t,\cdot)$ is a continuous piecewise affine function.*

Steps of the proof

1. We first solve the super-hedging problem between two dates $t-1$ and t
2. We define V^*_{t-1} as the infimum of all the superhedging prices at time $t-1$ over all integer-valued strategies in $\mathbb{Z}$, i.e. We denote

$$V^*_{t-1} := \operatorname*{ess\,inf}_{\theta_{t-1}\in\mathbb{L}^0(\mathbb{Z},\mathcal{F}_{t-1})} V_{t-1}(\theta_{t-1}).$$

3. We show that $V^*_{t-1} = g_{t-1}(S_{t-1})$ where g_{t-1} is a continuous piecewise linear function.
4. We show that the procedure may be propagated backwardly and deduce the minimal super-hedging price at time $t=0$ by induction.

Let's speak in vector Lattices language.

Some RS preliminaries

- (Kuo, Labuschagne and Watson)
  - A conditional expectation on a Dedekind complete Riesz space E is a positive order continuous projection $T : E \to E$, with range $R(T)$, a Dedekind complete Riesz subspace of E, satisfying $Te = e$ where e is a weak order unit.
  - For any conditional expectation T on E, there exists a largest Riesz subspace of the universal completion E^u of E, called the natural domain of T and denoted by $L^1(T)$, to which T extends uniquely to a conditional expectation.
  - A filtration on a Dedekind complete Riesz space E is a family of conditional expectations $(T_i)_{i\in\mathbb{N}}$, on E with $T_iT_j = T_jT_i = T_i$ for all $j > i$.

Some RS preliminaries

- (Azouzi and Trabelsi)
  - $L^\infty(T) = \{f \in L^1(T) : \exists\, g \in R(T),\ |f| \leq g\}$.
  - The space $L^\infty(T)$ is equipped with a vector valued norm given by $||f||_{\infty,T} = \inf\{g \in R(T) : |f| \leq g\}$.
  - $R(T)L^1(T) \subset L^1(T)$.
  - $L^\infty(T)$ is an f-subalgebra of $L^1(T)^u$.

Dictionary

$(\Omega, \mathcal{F}, \mathbb{P})$: probability space	E: Dedekind complete Riesz space.
X	$f \in E$
$A \subset \Omega$: event	P: order projection
$\mathbb{E}$: expectation	T: conditional expectation
χ_A: indicator function	p: component of the weak order unit e

Definition (A-B-C-M)

Let E be a Dedekind complete Riesz space with weak order unit $e > 0$ equipped with a conditional expectation T and f in $L^\infty(T)$. The conditional essential supremum of f with respect to T is given by

$$M_T(f) = \inf\{g \in R(T), f \leq g\}.$$

Similarly, we define the conditional essential infimum of f with respect to T in $L^\infty(T)$ as follows

$$m_T(f) = \sup\{g \in R(T), g \leq f\}.$$

Riesz space model of the financial market (A-B-C-M)

- A filtration $(\mathbb{F}_t)_{t\in[|0,T|]}$ where T is the time horizon.
- d risky assets.
- Prices process: $(S_t)_{t\in[0,T]}$ such that $S_t \in R(\mathbb{F}_t)^+$.
- Quantities process: $(\theta_t)_{t\in[0,T]}$ such that $\theta_t \in R(\mathbb{F}_t)$, called strategy.
- The portfolio values at time t: $V_t = S_t\theta_t = \sum_{k=1}^{d} \theta_t^k S_t^k$.
- A self-financing discrete time portfolio process $(V_t)_{0\le t\le T}$ satisfies:

$$\Delta V_t = V_t - V_{t-1} = \theta_{t-1}\Delta S_t, \quad t = 1, \cdots, T.$$

And its terminal value of at time T, starting at time t, is then

$$V_{t,T} = \sum_{i=t}^{T} \Delta V_i = \sum_{i=t}^{T} \theta_{i-1}\Delta S_i.$$

For $t \in [0, T]$ we denote by $\mathcal{V}_{t,T}$ the set of all T terminal elementary portfolios, starting with zero initial capital at time t :

$$\mathcal{V}_{t,T} = \left\{ v_{t,T} = \sum_{i=t}^{T} \theta_{i-1} \Delta S_i, \theta_i \in R(\mathbb{F}_i) \right\}.$$

Definition

An amount of money (or, contingent claim in the financial vocabulary) $h_T \in R(\mathbb{F}_T)$ is said to be super-hedgeable at time t if there exists $p_t \in R(\mathbb{F}_t)$ such that starting with this initial capital, one can find a portfolio strategy $v_{t,T} \in \mathcal{V}_{t,T}$ allowing him to exceed h_T at time T, which can be expressed as follows:

$$p_t + v_{t,T} \geq h_T.$$

We say that p_t is a price for h_t.

The set of all super-hedgeable claims with zero initial endowment at time t is then given by

$$A_{t,T} = \left\{v_{t,T} - \varepsilon_T, v_{t,T} \in V_{t,T}, \varepsilon_T \in R(\mathbb{F}_T)^+\right\}.$$

Let $P_{t,T}(h_T)$ be the set of all super-hedging prices $p_t \in R(\mathbb{F}_t)$ at time t for the contingent claim h_T.
The minimal super-hedging price is

$$\pi_{t,T}(h_T) := m_{\mathbb{F}_t}(P_t(h_T)).$$

We denote by $P_{t,T} = P_{t,T}(0)$ the set of all super-hedging prices for the zero claim, precisely

$$P_{t,T} = \left\{p_t \in R(\mathbb{F}_t) : \text{ there exists } v_{t,T} \in \mathcal{V}_{t,T} : p_t + v_{t,T} \geqslant 0\right\}.$$

Lemma

We have $P_{t,T} = (-A_{t,T}) \cap R(\mathbb{F}_t)$ *and*

$$P_{t,T} = \left\{M_{\mathbb{F}_t}(-v_{t,T}) : v_{t,T} \in \mathcal{V}_{t,T}\right\} + R(\mathbb{F}_t)^+.$$

AIP

We say that an immediate profit holds at time $t < T$ if it is possible to have a positive portfolio value from a negative initial price at time t.
On the contrary, we say that the Absence of Immediate Profit (AIP) holds, that is,

$$P_{t,T} \cap R(\mathbb{F}_t)^- = A_{t,T} \cap R(\mathbb{F}_t)^+ = \{0\}, \text{ for all } t < T.$$

Lemma

The following statements are equivalent for a financial market.

(i) *The AIP condition holds;*

(ii) *For any* $t \leq T$ *and for any* $v_{t,T} \in \mathcal{V}_{t,\mathbb{F}}$, $m_{\mathbb{F}_t}(v_{t,\mathbb{F}}) \leqslant 0$.

(iii) $\pi_{t,T}(0) = m_{\mathbb{F}_t}(P_t(0)) = 0$.

(iv) $P_{t,T} = R(\mathbb{F}_t)^+$.

Theorem (A-B-C-M)

The AIP condition holds if and only if, for all $0 \leq t < T$,

$$m_{\mathbb{F}_t}(S_{t+1}) \leq S_t \leq M_{\mathbb{F}_t}(S_{t+1}).$$

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Thank you.