# A functional representation approach to vector lattice covers for spaces of compact operators

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# Motivation (1)

Let $X$, $Y$ be Archimedean vector lattices and consider the space $\mathrm{L}^{\mathrm{r}}(X, Y)$ of regular linear operators with the cone $\mathrm{L}_{+}(X, Y) := \{T \in \mathrm{L}(X, Y);\, T[X_{+}] \subseteq Y_{+}\}$.

- If $Y$ is Dedekind complete:
  $\mathrm{L}^{\mathrm{r}}(X, Y)$ is a Dedekind complete vector lattice
- In general:
  $\mathrm{L}^{\mathrm{r}}(X, Y)$ is an Archimedean directed ordered vector space

  Problem: Determine the set of operators that have a modulus.

# Motivation (2)

Let $Z$ be an Archimedean directed ordered vector space.
Determine $M := \{z \in Z;\ |z| := \sup\{z, -z\} \text{ exists}\}$.

## Proposition (K., Stennder, van Gaans, 2021)

*For $z \in Z$, the following are equivalent.*

(i) $|z|$ *exists.*

(ii) *There are $z_1, z_2 \in Z_+$ with $z_1 \perp z_2$ and $z = z_1 - z_2$.*

- We are going to define disjointness in ordered vector spaces.
- We use a vector lattice cover of $Z$ to determine all disjoint elements (and, hence, all elements with modulus).

# Ordered vector spaces

Let $X$ be a (real) vector space. A partial order $\leq$ on $X$ is called a vector space order if

(a) $x, y, z \in X$ and $x \leq y$ imply $x + z \leq y + z$,

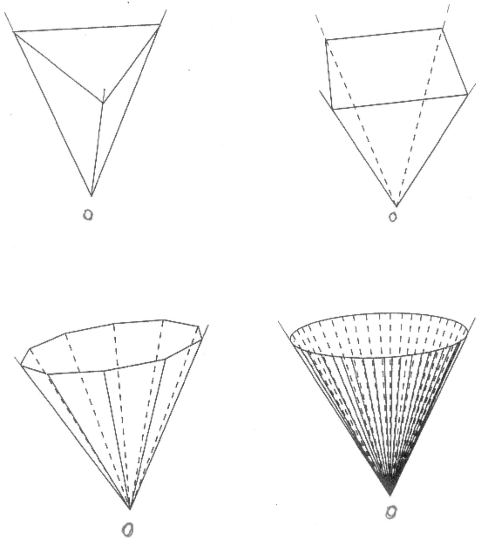
(b) $x \in X$, $0 \leq x$ and $\lambda \in [0, \infty)$ imply $0 \leq \lambda x$.

The set $X_+ := \{x \in X; 0 \leq x\}$ is then a cone in $X$, i.e., $x, y \in X_+$, $\lambda \in [0, \infty)$ imply $\lambda x + y \in X_+$, and $X_+ \cap (-X_+) = \{0\}$.

$X$ is then called an ordered vector space (ovs).

For $X, Y$ being ovs and $T \colon X \to Y$ a linear operator, $T$ is called bipositive if $T$ is positive (i.e., $T[X_+] \subseteq Y_+$) and, for $x \in X$, $Tx \in Y_+$ implies $x \in X_+$.

# Cones in $\mathbb{R}^3$ – from vector lattice to anti-lattice



# Pre-Riesz spaces

An ordered vector spaces $X$ is called a pre-Riesz space if there exist a vector lattice $Y$ and a bipositive linear map $i\colon X \to Y$ such that $i[X]$ is order dense in $Y$, i.e., for every $y \in Y$ one has

$$y = \inf\{i(x);\ x \in X,\ i(x) \geq y\}.$$

$(Y, i)$ is called a vector lattice cover.

Every vector lattice is a pre-Riesz space.

An ovs $X$ is called Archimedean if, for every $x, y \in X$ such that $nx \leq y$ for all $n \in \mathbb{N}$, one has that $x \leq 0$.
$X$ is directed if and only if $X_+$ is generating, i.e., $X = X_+ - X_+$.

## Proposition

- *Every Archimedean directed ovs is a pre-Riesz space.*
- *Every pre-Riesz space is directed.*

If $X$, $Y$ are ovs such that $X$ is directed and $Y$ is Archimedean, then $\mathrm{L}_+(X, Y) := \{T\colon X \to Y;\ T \text{ linear},\ T[X_+] \subseteq Y_+\}$ is a cone and $\mathrm{L}^{\mathrm{r}}(X, Y) := \mathrm{L}_+(X, Y) - \mathrm{L}_+(X, Y)$ is an Archimedean directed ovs, hence pre-Riesz.

Explicite construction of vector lattice covers of spaces of operators:

- $\mathrm{L}^{\mathrm{r}}(\ell_0^\infty, Y)$, where $\ell_0^\infty$ is the space of all finally constant sequences and $Y$ is an Archimedean vector lattice [Wickstead, 2024]
- generalized version in [Starkey, Xanthos, 2025]

Hereby, the vector lattice cover is a space of operators, where the range space is Dedekind complete.

- The operator norm closure $\mathcal{C}(X, Y)$ of the finite rank operators within $\mathcal{L}(X, Y)$, where $X$ and $Y$ are appropriate ordered normed spaces such that $\mathcal{L}(X, Y)_+$ contains an order unit [van Gaans, Glück, K., 2025]

Vector lattice cover is $\mathrm{C}(\Omega)$ for $\Omega$ being a compact Hausdorff space.

# Order unit spaces

Let $X$ be an Archimedean ovs with order unit $u$, i.e., for every $x \in X$ there is a $\lambda \in (0, \infty)$ such that $-\lambda u \leq x \leq \lambda u$. By defining

$$\|\cdot\|_u : X \to [0, \infty), \quad x \mapsto \|x\|_u := \inf\{\lambda \in (0, \infty);\ -\lambda u \leq x \leq \lambda u\},$$

$X$ is a normed space. $X'$ denotes the (norm) dual space of $X$ and $X'_+ := \{\varphi \in X';\ \varphi[X_+] \subseteq [0, \infty)\}$ the dual cone. The set

$$\Sigma = \{\varphi \in X'_+;\ \varphi(u) = 1\}$$

is a weakly-$*$ compact base of $X'_+$.

Note: Order unit spaces are directed.

# Kadison's representation

Define

$$\Psi\colon X \to \mathrm{C}(\Sigma), \qquad x \mapsto (\psi \mapsto \psi(x)).$$

- $\Psi$ is linear, bipositive, and maps $u$ to the constant-1 function.
- For every $x \in X$, the function $\Psi(x)$ is affine on $\Sigma$.

To obtain an order dense embedding, one has to consider

$$\Phi\colon X \to \mathrm{C}\left(\overline{\mathrm{ext}(\Sigma)}\right), \qquad x \mapsto (\psi \mapsto \psi(x)).$$

[K.,Lemmens, van Gaans, 2013]

$\left(\mathrm{C}\left(\overline{\mathrm{ext}(\Sigma)}\right), \Phi\right)$ is a vector lattice cover of $X$.

Our assumptions:

Let $X$, $Y$ be (non-zero) ordered normed spaces with closed cones. This implies that $X$ and $Y$ are Archimedean.

Consider the space $\mathcal{L}(X, Y)$ of linear norm bounded operators.

Let $X_+$ be total, i.e., $\overline{X_+ - X_+} = X$.
$X$ is total if and only if $\mathcal{L}(X, Y)_+$ is a cone.

Every $U$ in the interior of $\mathcal{L}(X, Y)_+$ is an order unit.

## Theorem (van Gaans, Glück, K., 2025)

*The following are equivalent:*

1. $\mathcal{L}(X, Y)_+$ *has non-empty interior in* $\mathcal{L}(X, Y)$.
2. *The cone* $Y_+$ *has non-empty interior in* $Y$ *and there exists an equivalent norm on* $X$ *which is additive on* $X_+$.

In this case, the interior of $\mathcal{L}(X, Y)_+$ contains a rank-1 operator:

For every interior point $y_0$ of $Y_+$ and every interior point $x_0'$ of $X_+'$, the operator $y_0 \otimes x_0'$ is an interior point of $\mathcal{L}(X, Y)_+$.

# Disjointness

For $x, y \in X_+$, define $x \perp y$ whenever $x \wedge y = 0$.

If $X$ is a vector lattice: For $x, y \in X$, $x \perp y$ whenever $|x| \wedge |y| = 0$.
Equivalent: $|x + y| = |x - y|$.

If $X$ is an ovs [van Gaans, K. 2006]:

$$x \perp y \quad :\Longleftrightarrow \quad \{x + y, -(x + y)\}^{\mathrm{u}} = \{x - y, -(x - y)\}^{\mathrm{u}}$$

For $M \subseteq X$, $M^{\mathrm{d}}$ denotes the disjoint complement of $M$.
$M$ is called a band, if $M = M^{\mathrm{dd}}$.

### Proposition (K., Stennder, van Gaans, 2021)

*Let $Z$ be a pre-Riesz space. The set of elements in $Z$ that possess a modulus in $Z$ equals*

$$\bigcup_{B \subseteq Z \text{ band}} \left( B_+ - B^{\mathrm{d}}_+ \right).$$

# Disjointness under embedding

**Proposition (van Gaans, K., 2006)**

*Let $X$ be a pre-Riesz space and $(Y, i)$ a vector lattice cover of $X$. Then one has, for every $x, y \in X$,*

$$x \perp y \iff i(x) \perp i(y).$$

Order denseness is needed for '$\Longrightarrow$'.

If $Y = \mathrm{C}(\Omega)$, disjointness is pointwise, and it is sufficient to consider a dense subset of $\Omega$.

Modification of Kadison's embedding:

## Theorem (van Gaans, Glück, K., 2025)

*Let $Z \neq \{0\}$ be an ordered normed space whose cone $Z_+$ has an interior point $z_0$ and let $S \subseteq Z'$ be a subset with the following properties:*

1. *One has $\langle s, z_0 \rangle = 1$ for all $s \in S$.*
2. *Every element of $S$ is an extremal vector of $Z'_+$.*
3. *The set $S$ determines positivity (i.e., for every $z \in Z$, one has $z \geq 0$ whenever $s(z) \geq 0$ for all $s \in S$).*

*Endow the weak-$*$ closure $\overline{S}$ with the weak-$*$ topology and consider*

$$\Phi \colon X \to \mathrm{C}(\overline{S}), \qquad x \mapsto (\psi \mapsto \psi(x)).$$

*Then $(\mathrm{C}(\overline{S}), \Phi)$ is a vector lattice cover of $Z$.*

$z_1 \perp z_2$ in $Z \iff$ for every $s \in S$, one has $s(z_1) = 0$ or $s(z_2) = 0$.

Question/Problem: Extremals in the cone of operators ??

$\mathcal{C}(X, Y)$ - closure of the space of finite rank operators in $\mathcal{L}(X, Y)$.

For $x \in X$ and $y' \in Y'$, define $y' \otimes x \in \mathcal{C}(X, Y)'$ by

$$(y' \otimes x)(T) := y'(Tx)$$

for all $T \in \mathcal{C}(X, Y)$.

## Theorem (vG, Gl, K., 2025)

*Let $X$ and $Y$ be ordered normed spaces with the following properties:*

1. *The cone $X_+$ is total and normal, and every extremal vector $x$ of $X_+$ is also extremal in the bidual cone $X''$.*
2. *The cone $Y_+$ is total.*

*For non-zero vectors $x \in X_+$ and $y' \in Y'_+$, the functional $y' \otimes x \in \mathcal{C}(X,Y)'_+$ is extremal in $\mathcal{C}(X,Y)'_+$ if and only if $x$ is extremal in $X_+$ and $y'$ is extremal in $Y'_+$.*

## Theorem (vG, Gl, K, 2025)

*Let $X$, $Y$ be (non-zero) ordered normed spaces such that:*

1. *The cone $X_+$ is total, there exists an equivalent norm on $X$ that is additive on $X_+$, and every extremal vector of $X_+$ is also extremal in the bidual cone $X''_+$. Moreover, the convex hull of the extremal vectors in $X_+$ is dense in $X_+$.*
2. *The cone $Y_+$ has non-empty interior.*

*For an interior point $y_0$ of $Y_+$ and an interior point $x'_0$ of $X'_+$, define*

$$S := \{ y' \otimes x : \ x \text{ is extremal in } X_+, \ y' \text{ is extremal in } Y'_+, \text{ and } \langle y', y_0 \rangle \langle x'_0, x \rangle = 1 \} \subseteq \mathcal{C}(X, Y)'.$$

*Then $(\mathrm{C}(\overline{S}), \Phi)$ is a vector lattice cover of $\mathcal{C}(X, Y)$.*

Examples:

- If $X$, $Y$ are finite-dimensional ordered vector spaces with closed and generating cones, then the assumptions of the theorem are satisfied.
  For polyhedral cones: [Schneider, Vidyasagar, 1970]
- The sequence space $X := \ell^1$ with its usual norm and the componentwise order satisfies assumption 1 of the theorem.
- Let $H$ be a Hilbert space. The space $X$ of self-adjoint trace class operators on $H$, endowed with the *Loewner cone* of operators $A \in X$ that satisfy $(v|Av) \geq 0$ for all $v \in H$, satisfies assumption 1 of the theorem.
  The extremal vectors of $X_+$ are precisely the strictly positive multiples of the rank-1 projections on $H$.

- Let $X$ be a real Banach space, let $x_0 \in X$ and $x_0' \in X'$ be such that $\langle x_0', x_0 \rangle = 1$ and endow $X$ with the *centered cone*

$$X_+ := \{x + rx_0 \colon\ x \in \ker x_0',\ r \geq \|x\|\}.$$

  Then $X_+$ is a closed cone with non-empty interior and there exists an equivalent norm on $X$ that is additive on $X_+$ [Glueck, 2016].

  If the space $X$ is reflexive, then $X$ satisfies assumption 1 of the theorem.

## Corollary

*In the setting of the above theorem, two operators* $T_1, T_2 \in \mathcal{C}(X, Y)$ *are disjoint if and only if, for all* *extremal vectors* $x \in X_+$ *and* $y' \in Y'_+$*, one has* $\langle y', T_1 x\rangle = 0$ *or* $\langle y', T_2 x\rangle = 0$.

- Bands in $\mathcal{C}(X, Y)$ are characterized by bisaturated subsets of $\overline{S}$.
- The set of elements in $\mathcal{C}(X, Y)$ that possess a modulus equals

$$\bigcup_{B \subseteq \mathcal{C}(X,Y) \text{ band}} \left(B_+ - B_+^{\mathrm{d}}\right).$$

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