Vector lattices with a view to applications in stochastics - I

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Partially ordered sets

- Let $\mathcal{A}$ be a non-empty set.
- We assume that there is a relation $\prec$ defined on $\mathcal{A}$.
- Recall that a **relation**, $\mathcal{R}$, is a subset of

$$\mathcal{A} \times \mathcal{A} = \{(x, y) \,|\, x, y \in \mathcal{A}\}.$$

  We write $x\mathcal{R}y$ if and only if $(x, y) \in \mathcal{R}$.
- We say that the relation $\prec$ is a **partial ordering** on $\mathcal{A}$ if it has the following three properties:

Reflexive: For each $x \in \mathcal{A}$, we have $x \prec x$.

Transitive: For each $x, y, z \in \mathcal{A}$ with $x \prec y$ and $y \prec z$, we have $x \prec z$.

Antisymmetric: If $x, y \in \mathcal{A}$ with $x \prec y$ and $y \prec x$, then $x = y$.

Examples of partially ordered set

- $\mathbb{R}$ with the relation $\leq$.
- If X is a set and $\mathcal{P}(X)$ denotes the power set of X, then **set containment**, $\subseteq$, is a partial ordering on $\mathcal{P}(X)$.
- $\mathbb{R}^n$ with **componentwise ordering**:

$$(x_1, \ldots, x_n) \prec (y_1, \ldots, y_n)$$

  if and only if

$$x_i \leq y_i, \quad \text{for all} \quad i = 1, \ldots, n.$$

- $\mathbb{R}^n$ with **lexicographic ordering**:

$$(x_1, \ldots, x_n) \prec (y_1, \ldots, y_n)$$

  if and only if, for some $j \in \{0, 1, \ldots, n\}$,

$$x_i = y_i, \quad \text{for all} \quad i \in I(j),$$

$$x_{j+1} < y_{j+1}, \quad \text{if} \quad j \leq n-1,$$

  where $I(j) = \{1, \ldots, j\}$ for $j \geq 1$ and $I(0) = \emptyset$.

Comparable elements

- Let $\mathcal{A}$ be a partially ordered set.
- We say that two elements x and y of $\mathcal{A}$ are **comparable** if either $x \prec y$ or $y \prec x$.
- If all elements of $\mathcal{A}$ are comparable with each other, then $\mathcal{A}$ is said to be **linearly ordered**.
- $\mathbb{R}$ is linearly ordered.
- $\mathbb{R}^n$ with componentwise ordering is not linearly ordered. For example consider $(0,1),(1,0) \in R^2$, then with componentwise ordering, $(0,1)$ and $(1,0)$ are not comparable.
- $\mathbb{R}^n$ with lexicographic ordering is linearly ordered. For example consider $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^2$. Then one of the following occurs: $x_1 < y_1$, then $(x_1,x_2) \prec (y_1,y_2)$; $x_1 > y_1$ then $(y_1,y_2) \prec (x_1,x_2)$; $x_1 = y_1$ and $x_2 \leq y_2$ giving $(x_1,x_2) \prec (y_1,y_2)$; or $x_1 = y_1$ and $x_2 > y_2$ so $(y_1,y_2) \prec (x_1,x_2)$.

Bounds

- Let X be a set with partial ordering $\prec$.
- Let A be a nonempty subset of X.
- If there is $x \in X$ so that $a \prec x$ for all $a \in A$, then x is said to an **upper bound** of A and A is said to be **bounded above.**
- Note that in the case of x being an upper bound of A, x needs to be comparable with every element of A.
- If x is an upper bound of A and for each upper bound y of A we have that $x \prec y$, then x is called the **least upper bound** of A or **supremum** of A.
- If the supremum of A exists, then it is **unique** (from the antisymmetry).
- Lower bounds and infimum can be analogously defined.

Maxima and Minima

- Let X be a set with partial ordering $\prec$.
- Let A be a nonempty subset of X.
- An element a of A is called **a maximal element** of A if ($a \prec x$ and $x \in A$) implies that $a = x$.
- If there exists $a \in A$ such that $x \prec a$ for all $x \in A$, then a is called the **largest element** of A.
- If a is the largest element of A, then a is also a maximal element of A and is the only maximal element of A.
- Similar definitions hold for minimal elements and smallest elements.

Examples of upper bounds, suprema and maxima

- We work in the partially ordered set $\mathbb{R}^2$ with componentwise ordering.
- Let $A = \{(1,0), (0,1)\}$.
- Both $(1,0)$ and $(0,1)$ are maximal elements of A. For example: if $(1,0) \prec (x_1, x_2) \in A$ then $(x_1, x_2) = (1,0)$. So $(1,0)$ is a maximal element.
- Neither $(1,0)$ nor $(0,1)$ is an upper bound of A.

Examples of upper bounds, suprema and maxima

- We work in the partially ordered set $\mathbb{R}^2$ with componentwise ordering.
- Let $A = \{(1,0),(0,1)\}$.
- $(1,1)$ is an upper bound for A.
- $(1,1)$ is not a maximal element of A as $(1,1) \notin A$.
- If (x_1, x_2) is an upper bound of A then, $(1,0) \prec (x_1, x_2)$ and $(0,1) \prec (x_1, x_2)$. Thus $(1,1) \prec (x_1, x_2)$, giving that $(1,1)$ is the least upper bound, or supremum, of A.

Ordered vector spaces (over $\mathbb{R}$)

- Let E be a vector space over $\mathbb{R}$.
- Assume that E is partially ordered by the relation $\prec$.
- We say that E is an **ordered vector space** if the following **compatibility conditions** between the partial ordering and the algebraic structure of E are satisfied:
  - If $x, y, z \in E$ with $x \prec y$, then $x + z \prec y + z$.
  - If $x, y \in E$ and $\alpha \in \mathbb{R}$ with $x \prec y$ and $\alpha \geq 0$, then $\alpha x \prec \alpha y$.
- We say that $x \in E$ is **positive** if $0 \prec x$.
- The **set of positive elements** of E will be denoted E^+.
- The notation $y \succ x$ is taken to mean the same as $x \prec y$.

Examples of ordered vector spaces

- $\mathbb{R}$ with the relation $\leq$.
- $\mathbb{R}^n$ with componentwise ordering.
- $\mathbb{R}^n$ with lexicographic ordering.
- $C(\Omega)$, the real valued continuous functions on a topological space Ω. The operations are defined pointwise:

$$f \leq g \quad \text{means} \quad f(\omega) \leq g(\omega) \quad \text{for all} \quad \omega \in \Omega,$$

$$(f+g)(\omega) = f(\omega) + g(\omega) \quad \text{for all} \quad \omega \in \Omega$$

$$(\alpha f)(\omega) = \alpha(f(\omega)) \quad \text{for all} \quad \omega \in \Omega$$

  with $f, g \in C(\Omega)$ and $\alpha \in \mathbb{R}$.
- $L^p(\Omega, \mathcal{A}, \mu)$, the a.e. equivalence classes of μ-measurable functions with p^{th} power of the absolute value of the function being integrable.
  The operations here are a.e. pointwise.

Lattices

- Let X be a partially ordered set, with partial ordering $\prec$.
- The partially ordered set X is a **lattice** if and only if

$$\text{the supremum of } x \text{ and } y, \text{ denoted } x \vee y$$

and

$$\text{the infimum of } x \text{ and } y, \text{ denoted } x \wedge y$$

exist in X for each $x, y \in X$.

Examples of lattices

- Let X be a non-empty set and $\mathcal{P}(X)$ denote the power set of X.
- As before we partially order $\mathcal{P}(X)$ by set containment.
- For $A, B \in \mathcal{P}(X)$ we have that $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.
- To see this, $A \cup B$ is an upper bound of $\{A, B\}$. If $C \in \mathcal{P}(X)$ with $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$, so $A \cup B$ is the least upper bound (supremum) of A and B.
- If Ω is a set, we take $(\mathcal{P}(X))^\Omega$ to be the maps from Ω to the power set of X with $F \prec G$ if $F(\omega) \subset G(\omega)$ for each $\omega \in \Omega$ is a lattice with

$$(F \vee G)(\omega) = F(\omega) \cup G(\omega)$$

and

$$(F \wedge G)(\omega) = F(\omega) \cap G(\omega)$$

for each $F, G \in (\mathcal{P}(X))^\Omega$.

Examples of lattices

- The set of real valued continuous function on $(-1,1)$, $C(-1,1)$, with pointwise operations is a lattice.
- $C^1(-1,1)$, the real valued continuous functions of $(-1,1)$ with continuous first derivative, is NOT a lattice under pointwise operation.
  To see this consider the 0 function (identically zero) and $f(x)=x$, then there is one least upper bound of $\{0,f\}$ (as it would require a corner at 0).
- The set $\{(0,0),(0,1),(2,0),(2,1)\}$ is a lattice (under componentwise ordering).

Riesz spaces

- A set X is a **Riesz space** if it has the following properties:
- It is an ordered vector space.
- With respect to the partial ordering, it is a lattice.
- Riesz spaces are also known as **vector lattices**.

Examples of Riesz spaces

- $\mathbb{R}$ with the relation $\leq$.
- $\mathbb{R}^n$ with componentwise ordering.
- $\mathbb{R}^n$ with lexicographic ordering.
- $C(X)$ the space of real valued continuous functions over a topological space X.
- $L^p(\Omega, \mathcal{A}, \mu)$ of a.e. equivalence classes with a.e. pointwise operations.
- $\mathbb{R}^X$ the space of all maps from a set X into $\mathbb{R}$ with pointwise operations and partial ordering. Here

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

for each $x \in X$ and $f, g \in \mathbb{R}^X$.

Lattice completeness

- Let X be a partially ordered set.
- X is said to be order complete if every non-empty subset of X has a supremum and an infimum.
- X is said to be Dedekind complete if every non-empty subset of X that is bounded above has a supremum.
- X is said to be σ-Dedekind complete if every non-empty countable subset of X that is bounded above has a supremum.
- Note that for X to be a lattice, it requires only that each set with two elements has a supremum.

Here endith the 1st lecture

Vector lattices with a view to applications in stochastics - II

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Riesz spaces

A set X is a **Riesz space** if it has the following properties:

- It is a vector space.
- It is a partially ordered set. We will denote this partial ordering by $\leq$.
- The algebraic and order structure are compatible.
- It is a lattice with respect to the partial ordering.

Cones

Let E be a Riesz space. We say that $F \subset E$ is a cone if:

- For each $a, b \in F$, we have $a + b \in F$.
- For each $a \in F$ and $\alpha \in \mathbb{R}_+$, we have $\alpha a \in F$.
- If $a \in F$ and $-a \in F$ then $a = 0$.

$E_+ := \{f \in E \,|\, 0 \leq f\}$ is a cone (called the positive cone of E).
Knowing E_+ is equivalent to knowing the partial ordering on E as

$$f \leq g \text{ if and only if } g - f \in E_+.$$

Positive decomposition

Let $f \in E$, where E is a Riesz space, then

- We denote
$$f^+ := f \vee 0,$$
$$f^- := (-f) \vee 0.$$
- Note that $f^+, f^- \in E_+$ with $f^+ \wedge f^- = 0$ (disjoint) and
$$f = f^+ - f^-.$$
- Thus
$$E = E^+ - E^+ = \{f - g \,|\, f, g \in E_+\}.$$
- We define
$$|f| := f \vee (-f),$$
then
$$|f| = f^+ + f^-.$$
- Further
$$f + g = (f \vee g) + (f \wedge g).$$

Riesz subspaces & ideals

Let E be a Riesz space.

- A vector subspace F of E is called a Riesz subspace if for each $f, g \in F$ we have that $f \vee g$ and $f \wedge g$ as computed in E are in F (in which case $f \vee g$ and $f \wedge g$ as computed in F).
- A subset S of E is said to be solid if for each $f \in E$ with $|f| \leq |g|$ for some $g \in S$, we have that $f \in S$.
- A solid vector subspace F of E is called an ideal of E.
- An ideal of E is also a Riesz subspace of E.

Example:

- If μ is a finite measure, then $L^\infty(\Omega, \mathcal{A}, \mu)$ is an ideal of $L^1(\Omega, \mathcal{A}, \mu)$.
- $C(\Omega)$ is a Riesz subspace of $\mathbb{R}^\Omega$, but is not an ideal of $\mathbb{R}^\Omega$.

Ideals

Let E be a Riesz space. $F \subset E$ is an ideal of E if and only if the following three conditions are met:

- F is a vector subspace of E.
- $f \in F$ if and only if $|f| \in F$.
- For each $g \in E_+$, we have that $g \wedge h \in F$ for each $0 \leq h \in F$.

Bands

Let E be a Riesz space and F be a subset of E. We say that F is a band in E if

- F is an ideal of E.
- If $D \subset F$ and $s := \sup D$ exists in E, then $s \in F$ (in which case s is also the supremum of D in F).

Example:

- If μ is a finite measure then $L^\infty(\Omega, \mathcal{A}, \mu)$ is an ideal but not a band of $L^1(\Omega, \mathcal{A}, \mu)$.
- In $\mathbb{R}^n$ with componentwise ordering, The set $\{x \in \mathbb{R}^n \,|\, x_i = 0, \forall i \in \Lambda\}$ is a band for each $\Lambda \subset \{1, \ldots, n\}$.

Directed sets

A non-empty set D is said to be directed, if it is equipped with a relation $\preceq$ which has the following properties:

Reflexive If $\alpha \in D$, then $\alpha \preceq \alpha$.

Transitive If $\alpha, \beta, \gamma \in D$ with $\alpha \preceq \beta$ and $\beta \preceq \gamma$ then $\alpha \preceq \gamma$.

Upper bd If $\alpha, \beta \in D$, then there exists $\gamma \in D$ with $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

Examples: $\mathcal{P}(X)$ is directed by set containment.
$\{(0,1),(1,0)\}$ with coordinatewise ordering is not a directed set.
$\{(0,1),(1,0),(1,2)\}$ with coordinatewise ordering is a directed set.

Nets

Let E be a Riesz space.

- Let D be a directed set.
- A family f_α indexed by $\alpha \in D$ is say to be a net in E.
- The net $f_\alpha, \alpha \in D$, is said to be increasing for each $\alpha \preceq \beta$ with $\alpha, \beta \in D$ and we have $f_\alpha \leq f_\beta$, denoted $f_\alpha \uparrow_{\alpha \in D}$. If $\sup_{\alpha \in D} f_\alpha$ exists and is g, then we write $f_\alpha \uparrow_{\alpha \in D} g$.
- The net $f_\alpha, \alpha \in D$, is said to be decreasing for each $\alpha \preceq \beta$ with $\alpha, \beta \in D$ and we have $f_\alpha \geq f_\beta$, denoted $f_\alpha \downarrow_{\alpha \in D}$. If $\inf_{\alpha \in D} f_\alpha$ exists and is h, then we write $f_\alpha \downarrow_{\alpha \in D} h$.

Sequences are special cases of nets (in the case $D = \mathbb{N}$).
One can also talk about nets being upwards directed or downwards directed, where the elements of the net are also the indicies.

Archimedean property

Let E be a Riesz space.

- If for each $f \in E_+$ the sequence $(\frac{1}{n}f)_{n\in\mathbb{N}}$ has $n^{-1}f \downarrow_{n\in\mathbb{N}} 0$, then the Riesz space E is said to be Archimedean.
- We will assume our spaces to be Archimedean - as this ensures uniqueness of order limits.

Order limits

Let E be an Archimedean Riesz space.

- Let $f_\alpha, \alpha \in D$, be a net in E.
- We say that the net $f_\alpha, \alpha \in D$, converges in order to f in E if:
- There is a net $g_\lambda, \lambda \in \Lambda$, with $g_\lambda \downarrow_{\lambda \in \Lambda} 0$ so that for each $\lambda \in \Lambda$ there is $\beta \in D$ such that

$$|f_\alpha - f| \leq g_\lambda, \quad \text{for all} \quad \beta \preceq \alpha.$$

- This is denoted as $f_\alpha \to f$, or if there are other notions of convergence available, then $f_\alpha \to_o f$.
- In the case of E being both Archimedean and Dedekind complete, we can choose that index set Λ to be the same as D.

Monotonic nets

- In the case of the net $f_\alpha, \alpha \in D$, being increasing with $f_\alpha \uparrow_{\alpha \in D} f$, we have that $f_\alpha \to_o f$.
- In the case of the net $f_\alpha, \alpha \in D$, being decreasing with $f_\alpha \downarrow_{\alpha \in D} f$, we have that $f_\alpha \to_o f$.

Order closure and Bands

Let E be an Archimedean Riesz space.

- A subset S of E is said to be order closed, if for each net f_α in S with order limit f in E, we have that $f \in S$.
- A band is an order closed ideal.

Here endith the 2^{nd} lecture

Vector lattices with a view to applications in stochastics - III

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A set X is a **Riesz space** if it has the following properties:

- It is a vector space.
- It is a partially ordered set. We will denote this partial ordering by $\leq$.
- The algebraic and order structure are compatible.
- It is a lattice with respect to the partial ordering.

Let E be a Riesz space.

- A vector subspace F of E is called a Riesz subspace if for each $f, g \in F$ we have that $f \vee g$ and $f \wedge g$ as computed in E are in F (in which case $f \vee g$ and $f \wedge g$ as computed in F).
- A subset S of E is said to be solid if for each $f \in E$ with $|f| \leq |g|$ for some $g \in S$, we have that $f \in S$.
- A solid vector subspace F of E is called an ideal of E.
- An ideal of E is also a Riesz subspace of E.
- An ideal F of E is an band if for each $D \subset F$ with $s := \sup D$ existing in E we have $s \in F$ (in which case s is also the supremum of D in F).

Principal ideals and order units

Let E be an Archimedean Riesz space.

- For each $f \in E$, we define
$$A_f := \{g \in E \mid \exists \lambda \in \mathbb{R}_+ \text{ with } |g| \leq \lambda |f|\}.$$
- A_f is called the ideal generated by f.
- An ideal generated by a single element of E is a called a principal ideal.
- If $e \in E_+$ and $A_e = E$, then e is said to be an order unit of E.

Examples: In L^∞, the constant 1 function is an order unit.
In $C(K)$, the real valued continuous functions on a compact set K, the positive constant functions are order units - in fact every Dedekind complete Archimedean Riesz space with order unit can be identified with a $C(K)$ space.

Principal bands and order units

Let E be an Archimedean Riesz space.

- For each $f \in E$ we define

$$B_f := \{g \in E \,|\, |g| \wedge (n|f|) \uparrow_{n\in\mathbb{N}} |g|\}.$$

- B_f is called the band generated by f.
- A band generated by a single element of E is a called a principal band.
- If $e \in E_+$ and $B_e = E$, then e is said to be a weak order unit of E.

Examples: In $L^1(\mu)$, where μ is a finite measure, the constant 1 function is a weak order unit.
Here, if $L^1(\mu)$ is infinite dimensional, then $L^1(\mu)$ does not have an order unit.

Extreme points

- Let S be a non-empty subset of a real vector space V.
- S is said to be convex, if for all $a, b \in S$ and $\lambda \in [0, 1]$ we have $a + \lambda(b - a) \in S$.
- If S is convex, we say that $c \in S$ is an extreme point of S if $c = a + \lambda(b - a)$ for some $a, b \in S$ and $0 < \lambda < 1$ implies $a = c = b$.

Components

Let E be an Archimedean Riesz space and $e \in E_+$.

- We say that $x \in E_+$ is a component of e if $x \leq e$ and

$$x \wedge (e - x) = 0.$$

- Let C_e denote the components of e in E.
- C_e is a Boolean algebra.
- C_e consists of the expreme points of the order interval

$$[0, e] := \{f \in E_+ \,|\, f \leq e\}.$$

Positive operators on Riesz spaces

Let E and F be a Archimedean Riesz spaces and $S : E \to F$ be a linear map with domain the whole of E.

- We say that S is a positive operator if $Sf \in F_+$ for each $f \in E_+$.
- We say that S is order continuous if for each net f_α convergent 0 in E we have that the net Sf_α is order convergent to Sf in F.
- We say that S is a Riesz homomorphism if

$$S(f \vee g) = Sf \vee Sg$$

for all $f, g \in E$.

Projection bands

Let E be an Archimedean Riesz space.

- A band B is said to be a projection band if
$$E = B \oplus B^d$$
where
$$B^d = \{f \in E \,|, |f| \wedge |g| = 0 \,\forall\, g \in B\}.$$
- In this case there is $P_B : E \to E$ a linear positive projection with range B and range of $I - P_B$ being B^d.
- If E is Dedekind complete, then every band is a projection band.

Principle bands

Let E be a Dedekind complete Riesz space.

- If f is in the positive cone, $E^+ := \{f \in E \mid f \geq 0\}$, of E then the band generated by f is given by

$$B_f = \{g \in E \ : \ |g| \wedge nf \uparrow_n |g|\}.$$

- Let P_f be the band projection onto B_f, then

$$\begin{aligned} P_f g &= P_f g^+ - P_f g^-, && \text{for} \quad g \in E, \\ P_f g &= \sup_{n=0,1,\dots} g \wedge nf, && \text{for} \quad g \in E^+. \end{aligned}$$

Principle bands and weak order units

Let E be a Dedekind complete Riesz space with weak order unit, say e.

- If B is a band in E, then B is a projection band, so we have that it is the range of a band projection, say P_B.
- Further B is a principle band with generator $P_B e$.
- The band generated by e is E.

Vector lattices with a view to applications in stochastics - IV

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Probability spaces

Definition

A probability space is a triple
$(\Omega, \mathcal{F}, P)$, where Ω is a set, $\mathcal{F}$ is a σ-algebra of subsets of Ω and P is a positive measure on $\mathcal{F}$ with $P(\Omega) = 1$.

Definition

A random variable is an element of real valued $L^1(\Omega, \mathcal{F}, P)$ and the expectation of the random variable f is

$$\mathbb{E}[f] = \int_\Omega f \, dP.$$

Conditioning

Definition

Let $A, B \in \mathcal{F}$, then the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Let $\{A_1, A_2, ..., A_n\}$ be a partition of Ω and Σ be the σ-algebra generated by this partition. Then the conditional expectation of f given Σ is the function with the value at x given by

$$\begin{aligned}
\mathbb{E}[f|\Sigma](x) &= \int_\Omega f\, dP(\cdot|A_i) \quad \text{where } x \in A_i \\
&= \frac{\int_{A_i} f\, dP}{P(A_i)} \quad \text{where } x \in A_i \\
&= \text{average of } f \text{ over } A_i.
\end{aligned}$$

Conditional expectations

More generally:
Let Σ be a sub-σ-algebra of $\mathcal{F}$. We define the conditional expectation of f with respect to Σ by

$$\mathbb{E}[f|\Sigma] = g,$$

where g is Σ-measurable and for each $A \in \Sigma$

$$\int_A g\, dP = \int_A f\, dP.$$

I.e. the average of g over A equals the average of f over A for each $A \in \Sigma$. The existence of f given Σ is ensured by the Radon-Nikodým theorem.

Consider $L^1(\Omega, \mathcal{F}, P)$, the **real** P-integrable functions over Ω. This is a vector space but with some additional stucture. If $f, g \in L^1(\Omega, \mathcal{F}, P)$, then we say that

$$f \leq g \quad \text{if} \quad f(x) \leq g(x) \quad \text{a.e. on } \Omega.$$

This order structure is compatible with the algebraic structure in that:

$$f \leq g \implies f + h \leq g + h$$

and

$$f \geq 0 \text{ and } 0 \leq \alpha \in \mathbb{R} \implies \alpha f \geq 0.$$

Definition

An ordered vector space is a pair $(V, \leq)$ where V is a real vector space with a partial ordering $\leq$ such that for $f, g, h \in V$:
1. if $f \leq g$ then $f + h \leq g + h$;
2. if $f \geq 0$ and $0 \leq \alpha \in \mathbb{R}$ then $\alpha f \geq 0$.

Hence $L^1(\Omega, \mathcal{F}, P)$ is an ordered vector space.
Also, if $f, g \in L^1(\Omega, \mathcal{F}, P)$ we can define functions $J, H \in L^1(\Omega, \mathcal{F}, P)$ by:

$$\begin{aligned} H(x) &= \max\{f(x), g(x)\} \\ J(x) &= \min\{f(x), g(x)\}. \end{aligned}$$

Supremum

We call H the supremum of f and g and denote it by $H = f \vee g$, and J the infimum of f and g and denote it $J = f \wedge g$. These concepts of supremum and infimum are compatible with the order structure on $L^1(\Omega, \mathcal{F}, P)$.

I.e. $f \le g \quad \textit{iff} \quad f \wedge g = f.$

Definition

Let $(V, \le)$ be an ordered vector space in which each pair of elements of V have both a $\sup(\vee)$ and $\inf(\wedge)$ compatible with $\le$. In this case we say $(V, \le)$ is a Riesz space.

Thus $L^1(\Omega, \mathcal{F}, P)$ is a Riesz space.

Positivity

Definition

A linear map $T : A \to B$, where A and B are Riesz spaces, is said to be positive if

$$f \geq 0 \implies Tf \geq 0.$$

Thus $\mathbb{E}[\,\cdot\,|\Sigma]$ is a positive linear projection.

Definition

g is a weak order unit in the Riesz space E if $g \geq 0$ and for all $f \geq 0$

$$f \wedge ng \uparrow_n f,$$

$$i.e. \quad \sup\{f \wedge ng | n \in \mathbb{N}\} = f.$$

As P is a finite measure, by LMCT, the constant function $\mathbf{1}$ is a weak order unit in $L^1(\Omega, \mathcal{F}, P)$ and $\mathbb{E}[\mathbf{1}|\Sigma] = \mathbf{1}$.

Weak order units

Also, if $g > 0$ a.e. and $g \in L^1(\Omega, \mathcal{F}, P)$, then g is a weak order unit of $L^1(\Omega, \mathcal{F}, P)$. But then $\mathbb{E}[g|\Sigma] > 0$ a.e. and so $\mathbb{E}[g|\Sigma]$ is again a weak order unit.
Hence $L^1(\Omega, \mathcal{F}, P)$ is a Riesz space with weak order unit and

$$\mathbb{E}[w.o.u.|\Sigma] = w.o.u..$$

Definition

Let $T : E \to F$ be a positive linear operator between Riesz spaces. T is order continuous if for each family $f_\alpha \downarrow 0$ we have $T(f_\alpha) \downarrow 0$.

From LDCT, $\mathbb{E}[\cdot|\Sigma]$ is order continuous.

Properties

The mapping of f to its conditional expectation has the following important properties, which give the key to our reformulation in terms of Riesz spaces:

1. $f \mapsto \mathbb{E}[f|\Sigma]$ is linear;
2. if $f \geq 0$ then $\mathbb{E}[f|\Sigma] \geq 0$ (is positive);
3. $\mathbb{E}[\mathbf{1}|\Sigma] = \mathbf{1}$ (preserves the 1 function);
4. $\mathbb{E}[\mathbb{E}[f|\Sigma]|\Sigma] = \mathbb{E}[f|\Sigma]$ (is idempotent);
5. if $f_n \uparrow f$ in $L^1(\Omega, \mathcal{F}, P)$ then $\mathbb{E}[f_n|\Sigma] \uparrow \mathbb{E}[f|\Sigma]$ in $L^1(\Omega, \Sigma, P)$ (is order continuous).

Together 1 and 4 give that $f \mapsto \mathbb{E}[f|\Sigma]$ is a projection. To make use of 2, 3 and 5 we develop the basics of Riesz space theory.

Probability spaces

- Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, i.e. Ω is a set, $\mathcal{F}$ is a σ-algebra of subsets of Ω and μ is a positive measure on $\mathcal{F}$ with $\mu(\Omega) = 1$.
- A random variable is an element of real valued $L^1(\Omega, \mathcal{F}, \mu)$.
- The expectation of the random variable f is

$$\mathbb{E}[f] = \int_\Omega f \, d\mu.$$

- Here $\mathbb{E}[f]$ can also be considered as the class of measurable functions on Ω with value a.e. $\int_\Omega f \, d\mu$ making

$$\mathbb{E} : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \mathcal{N}, \mu) \subset L^1(\Omega, \mathcal{F}, \mu),$$

where $\mathcal{N}$ consists of the sets of measure zero and their complements.

Conditional expectations

- For Σ a sub-σ-algebra of $\mathcal{F}$, the conditional expectation of f with respect to Σ, denoted $\mathbb{E}[f|\Sigma]$, is F where F is Σ-measurable and for each $A \in \Sigma$

$$\int_A F \, d\mu = \int_A f \, d\mu. \tag{1}$$

- We observe that

$$\mathbb{E}[\cdot|\mathcal{F}] : L^1(\Omega, \mathcal{F}, \mu) \to L^1(\Omega, \Sigma, \mu) \subset L^1(\Omega, \mathcal{F}, \mu)$$

is a positive order continuous projection onto $L^1(\Omega, \Sigma, \mu)$, which maps a.e. constant functions to a.e. constant functions.

Douglas-Andô-Rao-Sidak

The following result relates contractive projections to conditional expectations.

- Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $1 \leq p < \infty$.
- If $T : L^p(\Omega, \mathcal{F}, \mu) \to L^p(\Omega, \mathcal{F}, \mu)$ is a positive contractive projection with $T\mathbf{1} = \mathbf{1}$.
- Then $Tf = \mathbb{E}[f \mid \Sigma], f \in L^p(\Omega, \mathcal{F}, \mu)$, for a unique σ-algebra $\Sigma \subset \mathcal{F}$.
- The precise version quoted here is from Rao and attributed to Sidak.

Douglas-Andô

- In the works of Douglas [1] and Andô [2] the range spaces of contractive projections on L_p spaces over probability measures are characterized as conditional expectation operators.
- In particular Douglas characterizes the ranges spaces of conditional expectation operators as the range spaces of contractive projections on L_p that are closed under order limits and are vector sub-lattices of L_p.
- The work of Douglas will be returned to later in the Riesz space setting.

[1] R.G. DOUGLAS, Contractive projections on an L_1 space, *Pacific J. Math.*, **15** (1965), 443-462.

[2] T. ANDÔ, Contractive projections in L_p spaces, *Pacific J. Math.*, **17** (1966), 391-405.

Riesz space conditional expectation operators

We[3] introduced the following definition for a conditional expectation operator in a Riesz space.

Definition

Let E be a Dedekind complete Riesz space with weak order unit. A positive order continuous projection T on E with range, $R(T)$, an order closed Riesz subspace of E, is called a conditional expectation if $T(e)$ is a weak order unit of E for each weak order unit e in E.

One can assume that T is strictly positive (T is a positive operator with $Tf \neq 0$ for $f \in E_+ \backslash \{0\}$), by working in a quotient space if necessary.

[3] W.-C. Kuo, C.C.A. Labuschagne, B.A. Watson, Discrete time stochastic processes on Riesz spaces, *Indag. Mathem.*, **15** (2004), 435-451.

Example

- If $E = L^1(\Omega, \mathcal{F}, \mu)$ is a probability space and Σ is a sub-σ-algebra of $\mathcal{F}$, then E is a Dedekind complete Riesz space with weak order unit $e = \mathbf{1}$ and

$$Tf = \mathbb{E}[f|\Sigma]$$

  is a Riesz space conditional expectation operator on E with $Te = e$.
- The converse is also true. If T is a Riesz space strictly positive conditional expectation operator on $E = L^1(\Omega, \mathcal{F}, \mu)$, where μ is a probability measure, then there is a sub-σ-algebra, Σ, of $\mathcal{F}$ so that $T = \mathbb{E}[\cdot|\Sigma]$.

Example - σ-finite processes

- Dellacherie and Meyer considered σ-finite processes in the context of martingale theory. [4].
- Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and let $(\Omega_i)_{i\in\mathbb{N}}$ be a μ-measurable partition of Ω into sets of finite positive measure.
- Let $\mathcal{A}_0$ be the sub-σ-algebra of $\mathcal{A}$ generated by $(\Omega_i)_{i\in\mathbb{N}}$. We take the Riesz space

$$E = \{f \in L^0(\Omega, \mathcal{A}, \mu) \,|\, f|_{\Omega_i} \in L^\infty(\Omega_i, \mathcal{A}_i, \mu|_{\mathcal{A}_i}$$

  where $\mathcal{A}_i = \{U \cap \Omega_i \,|\, U \in \mathcal{A}\}$.
- Here E is a Riesz space under a.e. operations.

[4] Sections 39, 42 and 43 of C. Dellacherie, P.-A. Meyer, *Probabilities and Potentials: B, Theory of Martingales,* North Holland Publishing Company, 1982.

Example - σ-finite processes

- For $f \in E$ define

$$Tf(\omega) := \frac{\int_{\Omega_i} f \, d\mu}{\mu(\Omega_i)}, \quad \text{for} \quad \omega \in \Omega_i. \tag{2}$$

- Then T is a Riesz space conditional expectation operator on E.
- For $f \in E \cap L^1(\Omega, \mathcal{A}, \mu)$, we have $Tf = \mathbb{E}[f|\mathcal{A}_0]$.
- But we have neither $L^1(\Omega, \mathcal{A}, \mu) \subset E$ nor $E \subset L^1(\Omega, \mathcal{A}, \mu)$.

T-universal extension of E

- Each Riesz space E with weak order unit and conditional expectation operator T admits a T-universal completion (the maximal Riesz space to which T can be extended as a conditional expectation operator).
- The T-universal completion of E is

$$\hat{E} = \text{dom}(T) - \text{dom}(T)$$

where

$$\text{dom}(T) := \{f \in E^u_+ | \exists \text{ net } f_\alpha \uparrow f \text{ in } E^u, (f_\alpha) \subset E_+, Tf_\alpha \text{ bdd in } E^u\}.$$

- Here E^u is the universal completion of E and E^u_+ is the positive cone of E^u.
- Here $\hat{E}$ is a Dedekind complete Riesz space containing E as and order dense subspace, and having weak order unit.
- The space $\hat{E}$ will be denoted $L^1(T)$.

Extension of T to $\hat{E}$

- Each Riesz space conditional expectation operator admits unique extension to a conditional expectation on the T-universal completion, $\hat{E}$, of E.
- If $f \in \hat{E}_+$ then there is a net (f_α) in E_+ with (Tf_α) bounded in the universal completion, E^u, of E, with $f_\alpha \uparrow f$.
- Then $Tf_\alpha \uparrow$ in E and as such has limit, which we denote $\hat{T}(f)$, in $\hat{E}$.
- We extend $\hat{T}$ to the whole of $\hat{E}$ by setting

$$\hat{T}f = \hat{T}f^+ - \hat{T}f^-, \quad f \in \hat{E}.$$

- It can now be verified that the extension $\hat{T} : \hat{E} \to \hat{E}$ of T is a conditional expectation operator on $\hat{E}$.

Example - σ-finite processes

- Here

$$\hat{E} = \left\{ f \in L^0(\Omega, \mathcal{A}, \mu) \,\middle|\, \int_{\Omega_i} |f| \, d\mu < \infty \text{ for all } i \in \mathbb{N} \right\},$$

which is characterized by $f|_{\Omega_i} \in L^1(\Omega, \mathcal{A}, \mu)$, for each $i \in \mathbb{N}$.

-

$$\hat{T}f(\omega) := \frac{\int_{\Omega_i} f \, d\mu}{\mu(\Omega_i)}, \quad \text{for} \quad \omega \in \Omega_i. \tag{3}$$

- E has weak order unit $e = 1$, the function identically 1 on Ω, which again is a weak order unit for E, but is not in general in $L^1(\Omega, \mathcal{A}, \mu)$.
- The range of the generalized conditional expectation operator T is

$$R(T) = \{ f \in L^0(\Omega, \mathcal{A}, \mu) \,|\, f \text{ a.e. constant on } \Omega_i, i \in \mathbb{N} \},$$

which is an algebra.

Commutation in $L^1(\Omega, \mathcal{A}, \mu)$

- For Σ a sub-σ-algebra of $\mathcal{F}$, the conditional expectation of f with respect to Σ, denoted $\mathbb{E}[f|\Sigma]$, is defined to be in $L^1(\Omega, \Sigma, \mu)$ with
-
$$\int_\Omega \chi_A \mathbb{E}[f|\Sigma]\, d\mu = \int_\Omega \chi_A f\, d\,u$$
for each $A \in \Sigma$. So by the uniqueness of the conditional expectation, $\mathbb{E}[\chi_A f|\Sigma] = \chi_A \mathbb{E}[f|\Sigma]$.

Commutation in Riesz spaces

- Let E be a Dedekind complete Riesz space with weak order unit, T a conditional expectation on E and B the band in E generated by $g \in R(T) \cap C_e$, with associated band projection P_g. Then $TP_g = P_gT$.
- Why?
- Let $f \in E_+$ then

$$P_gf = \sup_{n\in\mathbb{N}}(ng) \wedge f = o - \lim_{n\to\infty}(ng) \wedge f.$$

- But T is order continuous so

$$TP_gf = o - \lim_{n\to\infty} T[(ng) \wedge f].$$

- But as T is a positive operator

$$T[(ng) \wedge f] \leq (nTg) \wedge Tf] = (ng) \wedge Tf.$$

- So taking order limits gives

$$TP_gf \leq P_gTf.$$

Commutation in Riesz spaces

- Now $P_{e-g} = I - P_g$ so

$$Tf - TP_g f = TP_{e-g} f \leq P_{e-g} Tf = Tf - P_g Tf.$$

- Thus $P_g Tf \leq TP_g f$.
- Combining with the previous slide gives $P_g Tf = TP_g f$ for $f \in E_+$.
- For $f \in E$, decomposing f in to $f^{\pm}$, we have $P_g Tf^{\pm} = TP_g f^{\pm}$.
- Thus

$$P_g Tf = P_g Tf^+ - P_g Tf^- = TP_g f^+ - TP_g f^- = TP_g f.$$

- This provides the starting point of the averaging property of conditional expectation operators.

Thank you!