

Vector lattices with a view to applications in stochastics - I

Wenchi Kuo

University of the Witwatersrand
CoE-MaSS
NITheCS

Partially ordered sets

- Let \mathcal{A} be a non-empty set.
- We assume that there is a relation \prec defined on \mathcal{A} .
- Recall that a **relation**, \mathcal{R} , is a subset of

$$\mathcal{A} \times \mathcal{A} = \{(x, y) \mid x, y \in \mathcal{A}\}.$$

We write $x\mathcal{R}y$ if and only if $(x, y) \in \mathcal{R}$.

- We say that the relation \prec is a **partial ordering** on \mathcal{A} if it has the following three properties:

Reflexive For each $x \in \mathcal{A}$, we have $x \prec x$.

Transitive For each $x, y, z \in \mathcal{A}$ with $x \prec y$ and $y \prec z$, we have $x \prec z$.

Antisymmetric If $x, y \in \mathcal{A}$ with $x \prec y$ and $y \prec x$, then $x = y$.

Examples of partially ordered set

- \mathbb{R} with the relation \leq .
- If X is a set and $\mathcal{P}(X)$ denotes the power set of X , then **set containment**, \subseteq , is a partial ordering on $\mathcal{P}(X)$.
- \mathbb{R}^n with **componentwise ordering**:

$$(x_1, \dots, x_n) \prec (y_1, \dots, y_n)$$

if and only if

$$x_i \leq y_i, \quad \text{for all } i = 1, \dots, n.$$

- \mathbb{R}^n with **lexicographic ordering**:

$$(x_1, \dots, x_n) \prec (y_1, \dots, y_n)$$

if and only if, for some $j \in \{0, 1, \dots, n\}$,

$$x_i = y_i, \quad \text{for all } i \in I(j),$$

$$x_{j+1} < y_{j+1}, \quad \text{if } j \leq n - 1,$$

where $I(j) = \{1, \dots, j\}$ for $j \geq 1$ and $I(0) = \emptyset$.

Comparable elements

- Let \mathcal{A} be a partially ordered set.
- We say that two elements x and y of \mathcal{A} are **comparable** if either $x \prec y$ or $y \prec x$.
- If all elements of \mathcal{A} are comparable with each other, then \mathcal{A} is said to be **linearly ordered**.
- \mathbb{R} is linearly ordered.
- \mathbb{R}^n with componentwise ordering is not linearly ordered. For example consider $(0, 1), (1, 0) \in \mathbb{R}^2$, then with componentwise ordering, $(0, 1)$ and $(1, 0)$ are not comparable.
- \mathbb{R}^n with lexicographic ordering is linearly ordered. For example consider $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Then one of the following occurs: $x_1 < y_1$, then $(x_1, x_2) \prec (y_1, y_2)$; $x_1 > y_1$ then $(y_1, y_2) \prec (x_1, x_2)$; $x_1 = y_1$ and $x_2 \leq y_2$ giving $(x_1, x_2) \prec (y_1, y_2)$; or $x_1 = y_1$ and $x_2 > y_2$ so $(y_1, y_2) \prec (x_1, x_2)$.

- Let X be a set with partial ordering \prec .
- Let A be a nonempty subset of X .
- If there is $x \in X$ so that $a \prec x$ for all $a \in A$, then x is said to be an **upper bound** of A and A is said to be **bounded above**.
- Note that in the case of x being an upper bound of A , x needs to be comparable with every element of A .
- If x is an upper bound of A and for each upper bound y of A we have that $x \prec y$, then x is called the **least upper bound** of A or **supremum** of A .
- If the supremum of A exists, then it is **unique** (from the antisymmetry).
- Lower bounds and infimum can be analogously defined.

Maxima and Minima

- Let X be a set with partial ordering \prec .
- Let A be a nonempty subset of X .
- An element a of A is called a **maximal element** of A if $(a \prec x \text{ and } x \in A)$ implies that $a = x$.
- If there exists $a \in A$ such that $x \prec a$ for all $x \in A$, then a is called the **largest element** of A .
- If a is the largest element of A , then a is also a maximal element of A and is the only maximal element of A .
- Similar definitions hold for minimal elements and smallest elements.

Examples of upper bounds, suprema and maxima

- We work in the partially ordered set \mathbb{R}^2 with componentwise ordering.
- Let $A = \{(1, 0), (0, 1)\}$.
- Both $(1, 0)$ and $(0, 1)$ are maximal elements of A .
For example: if $(1, 0) \prec (x_1, x_2) \in A$ then $(x_1, x_2) = (1, 0)$.
So $(1, 0)$ is a maximal element.
- Neither $(1, 0)$ nor $(0, 1)$ is an upper bound of A .

Examples of upper bounds, suprema and maxima

- We work in the partially ordered set \mathbb{R}^2 with componentwise ordering.
- Let $A = \{(1, 0), (0, 1)\}$.
- $(1, 1)$ is an upper bound for A .
- $(1, 1)$ is not a maximal element of A as $(1, 1) \notin A$.
- If (x_1, x_2) is an upper bound of A then, $(1, 0) \prec (x_1, x_2)$ and $(0, 1) \prec (x_1, x_2)$.
Thus $(1, 1) \prec (x_1, x_2)$, giving that $(1, 1)$ is the least upper bound, or supremum, of A .

Ordered vector spaces (over \mathbb{R})

- Let E be a vector space over \mathbb{R} .
- Assume that E is partially ordered by the relation \prec .
- We say that E is an **ordered vector space** if the following **compatibility conditions** between the partial ordering and the algebraic structure of E are satisfied:
 - If $x, y, z \in E$ with $x \prec y$, then $x + z \prec y + z$.
 - If $x, y \in E$ and $\alpha \in \mathbb{R}$ with $x \prec y$ and $\alpha \geq 0$, then $\alpha x \prec \alpha y$.
- We say that $x \in E$ is **positive** if $0 \prec x$.
- The **set of positive elements** of E will be denoted E^+ .
- The notation $y \succ x$ is taken to mean the same as $x \prec y$.

Examples of ordered vector spaces

- \mathbb{R} with the relation \leq .
- \mathbb{R}^n with componentwise ordering.
- \mathbb{R}^n with lexicographic ordering.
- $C(\Omega)$, the real valued continuous functions on a topological space Ω . The operations are defined pointwise:

$$f \leq g \quad \text{means} \quad f(\omega) \leq g(\omega) \quad \text{for all} \quad \omega \in \Omega,$$

$$(f + g)(\omega) = f(\omega) + g(\omega) \quad \text{for all} \quad \omega \in \Omega$$

$$(\alpha f)(\omega) = \alpha(f(\omega)) \quad \text{for all} \quad \omega \in \Omega$$

with $f, g \in C(\Omega)$ and $\alpha \in \mathbb{R}$.

- $L^p(\Omega, \mathcal{A}, \mu)$, the a.e. equivalence classes of μ -measurable functions with p^{th} power of the absolute value of the function being integrable.

The operations here are a.e. pointwise.

- Let X be a partially ordered set, with partial ordering \preceq .
- The partially ordered set X is a **lattice** if and only if

the supremum of x and y , denoted $x \vee y$

and

the infimum of x and y , denoted $x \wedge y$

exist in X for each $x, y \in X$.

Examples of lattices

- Let X be a non-empty set and $\mathcal{P}(X)$ denote the power set of X .
- As before we partially order $\mathcal{P}(X)$ by set containment.
- For $A, B \in \mathcal{P}(X)$ we have that $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.
- To see this, $A \cup B$ is an upper bound of $\{A, B\}$. If $C \in \mathcal{P}(X)$ with $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$, so $A \cup B$ is the least upper bound (supremum) of A and B .
- If Ω is a set, we take $(\mathcal{P}(X))^\Omega$ to be the maps from Ω to the power set of X with $F \prec G$ if $F(\omega) \subset G(\omega)$ for each $\omega \in \Omega$ is a lattice with

$$(F \vee G)(\omega) = F(\omega) \cup G(\omega)$$

and

$$(F \wedge G)(\omega) = F(\omega) \cap G(\omega)$$

for each $F, G \in (\mathcal{P}(X))^\Omega$.

Examples of lattices

- The set of real valued continuous function on $(-1, 1)$, $C(-1, 1)$, with pointwise operations is a lattice.
- $C^1(-1, 1)$, the real valued continuous functions of $(-1, 1)$ with continuous first derivative, is NOT a lattice under pointwise operation.

To see this consider the 0 function (identically zero) and $f(x) = x$, then there is one least upper bound of $\{0, f\}$ (as it would require a corner at 0).

- The set $\{(0, 0), (0, 1), (2, 0), (2, 1)\}$ is a lattice (under componentwise ordering).

- A set X is a **Riesz space** if it has the following properties:
- It is an ordered vector space.
- With respect to the partial ordering, it is a lattice.
- Riesz spaces are also known as **vector lattices**.

Examples of Riesz spaces

- \mathbb{R} with the relation \leq .
- \mathbb{R}^n with componentwise ordering.
- \mathbb{R}^n with lexicographic ordering.
- $C(X)$ the space of real valued continuous functions over a topological space X .
- $L^p(\Omega, \mathcal{A}, \mu)$ of a.e. equivalence classes with a.e. pointwise operations.
- \mathbb{R}^X the space of all maps from a set X into \mathbb{R} with pointwise operations and partial ordering. Here

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

for each $x \in X$ and $f, g \in \mathbb{R}^X$.

Lattice completeness

- Let X be a partially ordered set.
- X is said to be order complete if every non-empty subset of X has a supremum and an infimum.
- X is said to be Dedekind complete if every non-empty subset of X that is bounded above has a supremum.
- X is said to be σ -Dedekind complete if every non-empty countable subset of X that is bounded above has a supremum.
- Note that for X to be a lattice, it requires only that each set with two elements has a supremum.

Here endith the 1st lecture

Vector lattices with a view to applications in stochastics - II

Wenchi Kuo

University of the Witwatersrand
CoE-MaSS
NITheCS

A set X is a **Riesz space** if it has the following properties:

- It is a vector space.
- It is a partially ordered set. We will denote this partial ordering by \leq .
- The algebraic and order structure are compatible.
- It is a lattice with respect to the partial ordering.

Let E be a Riesz space. We say that $F \subset E$ is a cone if:

- For each $a, b \in F$, we have $a + b \in F$.
- For each $a \in F$ and $\alpha \in \mathbb{R}_+$, we have $\alpha a \in F$.
- If $a \in F$ and $-a \in F$ then $a = 0$.

$E_+ := \{f \in E \mid 0 \leq f\}$ is a cone (called the positive cone of E).
Knowing E_+ is equivalent to knowing the partial ordering on E as

$$f \leq g \text{ if and only if } g - f \in E_+.$$

Positive decomposition

Let $f \in E$, where E is a Riesz space, then

- We denote

$$f^+ := f \vee 0,$$

$$f^- := (-f) \vee 0.$$

- Note that $f^+, f^- \in E_+$ with $f^+ \wedge f^- = 0$ (disjoint) and

$$f = f^+ - f^-.$$

- Thus

$$E = E^+ - E^+ = \{f - g \mid f, g \in E_+\}.$$

- We define

$$|f| := f \vee (-f),$$

then

$$|f| = f^+ + f^-.$$

- Further

$$f + g = (f \vee g) + (f \wedge g).$$

Riesz subspaces & ideals

Let E be a Riesz space.

- A vector subspace F of E is called a Riesz subspace if for each $f, g \in F$ we have that $f \vee g$ and $f \wedge g$ as computed in E are in F (in which case $f \vee g$ and $f \wedge g$ as computed in F).
- A subset S of E is said to be solid if for each $f \in E$ with $|f| \leq |g|$ for some $g \in S$, we have that $f \in S$.
- A solid vector subspace F of E is called an ideal of E .
- An ideal of E is also a Riesz subspace of E .

Example:

- If μ is a finite measure, then $L^\infty(\Omega, \mathcal{A}, \mu)$ is an ideal of $L^1(\Omega, \mathcal{A}, \mu)$.
- $C(\Omega)$ is a Riesz subspace of \mathbb{R}^Ω , but is not an ideal of \mathbb{R}^Ω .

Let E be a Riesz space. $F \subset E$ is an ideal of E if and only if the following three conditions are met:

- F is a vector subspace of E .
- $f \in F$ if and only if $|f| \in F$.
- For each $g \in E_+$, we have that $g \wedge h \in F$ for each $0 \leq h \in F$.

Let E be a Riesz space and F be a subset of E . We say that F is a band in E if

- F is an ideal of E .
- If $D \subset F$ and $s := \sup D$ exists in E , then $s \in F$ (in which case s is also the supremum of D in F).

Example:

- If μ is a finite measure then $L^\infty(\Omega, \mathcal{A}, \mu)$ is an ideal but not a band of $L^1(\Omega, \mathcal{A}, \mu)$.
- In \mathbb{R}^n with componentwise ordering, The set $\{x \in \mathbb{R}^n \mid x_i = 0, \forall i \in \Lambda\}$ is a band for each $\Lambda \subset \{1, \dots, n\}$.

Directed sets

A non-empty set D is said to be directed, if it is equipped with a relation \preceq which has the following properties:

Reflexive If $\alpha \in D$, then $\alpha \preceq \alpha$.

Transitive If $\alpha, \beta, \gamma \in D$ with $\alpha \preceq \beta$ and $\beta \preceq \gamma$ then $\alpha \preceq \gamma$.

Upper bd If $\alpha, \beta \in D$, then there exists $\gamma \in D$ with $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

Examples: $\mathcal{P}(X)$ is directed by set containment.

$\{(0, 1), (1, 0)\}$ with coordinatewise ordering is not a directed set.

$\{(0, 1), (1, 0), (1, 2)\}$ with coordinatewise ordering is a directed set.

Let E be a Riesz space.

- Let D be a directed set.
- A family f_α indexed by $\alpha \in D$ is said to be a net in E .
- The net $f_\alpha, \alpha \in D$, is said to be increasing for each $\alpha \preceq \beta$ with $\alpha, \beta \in D$ and we have $f_\alpha \leq f_\beta$, denoted $f_\alpha \uparrow_{\alpha \in D}$. If $\sup_{\alpha \in D} f_\alpha$ exists and is g , then we write $f_\alpha \uparrow_{\alpha \in D} g$.
- The net $f_\alpha, \alpha \in D$, is said to be decreasing for each $\alpha \preceq \beta$ with $\alpha, \beta \in D$ and we have $f_\alpha \geq f_\beta$, denoted $f_\alpha \downarrow_{\alpha \in D}$. If $\inf_{\alpha \in D} f_\alpha$ exists and is h , then we write $f_\alpha \downarrow_{\alpha \in D} h$.

Sequences are special cases of nets (in the case $D = \mathbb{N}$).

One can also talk about nets being upwards directed or downwards directed, where the elements of the net are also the indices.

Let E be a Riesz space.

- If for each $f \in E_+$ the sequence $(\frac{1}{n}f)_{n \in \mathbb{N}}$ has $n^{-1}f \downarrow_{n \in \mathbb{N}} 0$, then the Riesz space E is said to be Archimedean.
- We will assume our spaces to be Archimedean - as this ensures uniqueness of order limits.

Let E be an Archimedean Riesz space.

- Let $f_\alpha, \alpha \in D$, be a net in E .
- We say that the net $f_\alpha, \alpha \in D$, converges in order to f in E if:
- There is a net $g_\lambda, \lambda \in \Lambda$, with $g_\lambda \downarrow_{\lambda \in \Lambda} 0$ so that for each $\lambda \in \Lambda$ there is $\beta \in D$ such that

$$|f_\alpha - f| \leq g_\lambda, \quad \text{for all } \beta \preceq \alpha.$$

- This is denoted as $f_\alpha \rightarrow f$, or if there are other notions of convergence available, then $f_\alpha \rightarrow_o f$.
- In the case of E being both Archimedean and Dedekind complete, we can choose that index set Λ to be the same as D .

- In the case of the net $f_\alpha, \alpha \in D$, being increasing with $f_\alpha \uparrow_{\alpha \in D} f$, we have that $f_\alpha \rightarrow_o f$.
- In the case of the net $f_\alpha, \alpha \in D$, being decreasing with $f_\alpha \downarrow_{\alpha \in D} f$, we have that $f_\alpha \rightarrow_o f$.

Order closure and Bands

Let E be an Archimedean Riesz space.

- A subset S of E is said to be order closed, if for each net f_α in S with order limit f in E , we have that $f \in S$.
- A band is an order closed ideal.

Here endith the 2^{nd} lecture

Vector lattices with a view to applications in stochastics - III

Wenchi Kuo

University of the Witwatersrand
CoE-MaSS
NITheCS

A set X is a **Riesz space** if it has the following properties:

- It is a vector space.
- It is a partially ordered set. We will denote this partial ordering by \leq .
- The algebraic and order structure are compatible.
- It is a lattice with respect to the partial ordering.

Riesz subspaces, ideals & bands

Let E be a Riesz space.

- A vector subspace F of E is called a Riesz subspace if for each $f, g \in F$ we have that $f \vee g$ and $f \wedge g$ as computed in E are in F (in which case $f \vee g$ and $f \wedge g$ as computed in F).
- A subset S of E is said to be solid if for each $f \in E$ with $|f| \leq |g|$ for some $g \in S$, we have that $f \in S$.
- A solid vector subspace F of E is called an ideal of E .
- An ideal of E is also a Riesz subspace of E .
- An ideal F of E is a band if for each $D \subset F$ with $s := \sup D$ existing in E we have $s \in F$ (in which case s is also the supremum of D in F).

Principal ideals and order units

Let E be an Archimedean Riesz space.

- For each $f \in E$, we define

$$A_f := \{g \in E \mid \exists \lambda \in \mathbb{R}_+ \text{ with } |g| \leq \lambda|f|\}.$$

- A_f is called the ideal generated by f .
- An ideal generated by a single element of E is called a principal ideal.
- If $e \in E_+$ and $A_e = E$, then e is said to be an order unit of E .

Examples: In L^∞ , the constant 1 function is an order unit.

In $C(K)$, the real valued continuous functions on a compact set K , the positive constant functions are order units - in fact every Dedekind complete Archimedean Riesz space with order unit can be identified with a $C(K)$ space.

Principal bands and order units

Let E be an Archimedean Riesz space.

- For each $f \in E$ we define

$$B_f := \{g \in E \mid |g| \wedge (n|f|) \uparrow_{n \in \mathbb{N}} |g|\}.$$

- B_f is called the band generated by f .
- A band generated by a single element of E is called a principal band.
- If $e \in E_+$ and $B_e = E$, then e is said to be a weak order unit of E .

Examples: In $L^1(\mu)$, where μ is a finite measure, the constant 1 function is a weak order unit.

Here, if $L^1(\mu)$ is infinite dimensional, then $L^1(\mu)$ does not have an order unit.

Extreme points

- Let S be a non-empty subset of a real vector space V .
- S is said to be convex, if for all $a, b \in S$ and $\lambda \in [0, 1]$ we have $a + \lambda(b - a) \in S$.
- If S is convex, we say that $c \in S$ is an extreme point of S if $c = a + \lambda(b - a)$ for some $a, b \in S$ and $0 < \lambda < 1$ implies $a = c = b$.

Let E be an Archimedean Riesz space and $e \in E_+$.

- We say that $x \in E_+$ is a component of e if $x \leq e$ and

$$x \wedge (e - x) = 0.$$

- Let C_e denote the components of e in E .
- C_e is a Boolean algebra.
- C_e consists of the extreme points of the order interval

$$[0, e] := \{f \in E_+ \mid f \leq e\}.$$

Positive operators on Riesz spaces

Let E and F be a Archimedean Riesz spaces and $S : E \rightarrow F$ be a linear map with domain the whole of E .

- We say that S is a positive operator if $Sf \in F_+$ for each $f \in E_+$.
- We say that S is order continuous if for each net f_α convergent 0 in E we have that the net Sf_α is order convergent to Sf in F .
- We say that S is a Riesz homomorphism if

$$S(f \vee g) = Sf \vee Sg$$

for all $f, g \in E$.

Let E be an Archimedean Riesz space.

- A band B is said to be a projection band if

$$E = B \oplus B^d$$

where

$$B^d = \{f \in E \mid |f| \wedge |g| = 0 \forall g \in B\}.$$

- In this case there is $P_B : E \rightarrow E$ a linear positive projection with range B and range of $I - P_B$ being B^d .
- If E is Dedekind complete, then every band is a projection band.

Let E be a Dedekind complete Riesz space.

- If f is in the positive cone, $E^+ := \{f \in E \mid f \geq 0\}$, of E then the band generated by f is given by

$$B_f = \{g \in E : |g| \wedge nf \uparrow_n |g|\}.$$

- Let P_f be the band projection onto B_f , then

$$\begin{aligned} P_f g &= P_f g^+ - P_f g^-, & \text{for } g \in E, \\ P_f g &= \sup_{n=0,1,\dots} g \wedge nf, & \text{for } g \in E^+. \end{aligned}$$

Principle bands and weak order units

Let E be a Dedekind complete Riesz space with weak order unit, say e .

- If B is a band in E , then B is a projection band, so we have that it is the range of a band projection, say P_B .
- Further B is a principle band with generator $P_B e$.
- The band generated by e is E .

Vector lattices with a view to applications in stochastics - IV

Bruce A. Watson

University of the Witwatersrand
CoE-MaSS
NITheCS

Definition

A probability space is a triple (Ω, \mathcal{F}, P) , where Ω is a set, \mathcal{F} is a σ -algebra of subsets of Ω and P is a positive measure on \mathcal{F} with $P(\Omega) = 1$.

Definition

A random variable is an element of real valued $L^1(\Omega, \mathcal{F}, P)$ and the expectation of the random variable f is

$$\mathbb{E}[f] = \int_{\Omega} f \, dP.$$

Definition

Let $A, B \in \mathcal{F}$, then the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Let $\{A_1, A_2, \dots, A_n\}$ be a partition of Ω and Σ be the σ -algebra generated by this partition. Then the conditional expectation of f given Σ is the function with the value at x given by

$$\begin{aligned}\mathbb{E}[f|\Sigma](x) &= \int_{\Omega} f dP(\cdot|A_i) \quad \text{where } x \in A_i \\ &= \frac{\int_{A_i} f dP}{P(A_i)} \quad \text{where } x \in A_i \\ &= \text{average of } f \text{ over } A_i.\end{aligned}$$

Conditional expectations

More generally:

Let Σ be a sub- σ -algebra of \mathcal{F} . We define the conditional expectation of f with respect to Σ by

$$\mathbb{E}[f|\Sigma] = g,$$

where g is Σ -measurable and for each $A \in \Sigma$

$$\int_A g \, dP = \int_A f \, dP.$$

I.e. the average of g over A equals the average of f over A for each $A \in \Sigma$. The existence of f given Σ is ensured by the Radon-Nikodým theorem.

Consider $L^1(\Omega, \mathcal{F}, P)$, the **real** P -integrable functions over Ω . This is a vector space but with some additional structure. If $f, g \in L^1(\Omega, \mathcal{F}, P)$, then we say that

$$f \leq g \quad \text{if} \quad f(x) \leq g(x) \quad \text{a.e. on } \Omega.$$

This order structure is compatible with the algebraic structure in that:

$$f \leq g \implies f + h \leq g + h$$

and

$$f \geq 0 \text{ and } 0 \leq \alpha \in \mathbb{R} \implies \alpha f \geq 0.$$

Definition

An ordered vector space is a pair (V, \leq) where V is a real vector space with a partial ordering \leq such that for $f, g, h \in V$:

1. if $f \leq g$ then $f + h \leq g + h$;
2. if $f \geq 0$ and $0 \leq \alpha \in \mathbb{R}$ then $\alpha f \geq 0$.

Hence $L^1(\Omega, \mathcal{F}, P)$ is an ordered vector space.

Also, if $f, g \in L^1(\Omega, \mathcal{F}, P)$ we can define functions $J, H \in L^1(\Omega, \mathcal{F}, P)$ by:

$$H(x) = \max\{f(x), g(x)\}$$

$$J(x) = \min\{f(x), g(x)\}.$$

Supremum

We call H the supremum of f and g and denote it by $H = f \vee g$, and J the infimum of f and g and denote it $J = f \wedge g$. These concepts of supremum and infimum are compatible with the order structure on $L^1(\Omega, \mathcal{F}, P)$.

$$\text{i.e. } f \leq g \text{ iff } f \wedge g = f.$$

Definition

Let (V, \leq) be an ordered vector space in which each pair of elements of V have both a $\sup(\vee)$ and $\inf(\wedge)$ compatible with \leq . In this case we say (V, \leq) is a Riesz space.

Thus $L^1(\Omega, \mathcal{F}, P)$ is a Riesz space.

Definition

A linear map $T : A \rightarrow B$, where A and B are Riesz spaces, is said to be positive if

$$f \geq 0 \implies Tf \geq 0.$$

Thus $\mathbb{E}[\cdot | \Sigma]$ is a positive linear projection.

Definition

g is a weak order unit in the Riesz space E if $g \geq 0$ and for all $f \geq 0$

$$f \wedge ng \uparrow_n f,$$

$$\text{i.e. } \sup\{f \wedge ng | n \in \mathbb{N}\} = f.$$

As P is a finite measure, by LMCT, the constant function $\mathbf{1}$ is a weak order unit in $L^1(\Omega, \mathcal{F}, P)$ and $\mathbb{E}[\mathbf{1} | \Sigma] = \mathbf{1}$.

Also, if $g > 0$ a.e. and $g \in L^1(\Omega, \mathcal{F}, P)$, then g is a weak order unit of $L^1(\Omega, \mathcal{F}, P)$. But then $\mathbb{E}[g|\Sigma] > 0$ a.e. and so $\mathbb{E}[g|\Sigma]$ is again a weak order unit.

Hence $L^1(\Omega, \mathcal{F}, P)$ is a Riesz space with weak order unit and

$$\mathbb{E}[w.o.u.|\Sigma] = w.o.u..$$

Definition

Let $T : E \rightarrow F$ be a positive linear operator between Riesz spaces. T is order continuous if for each family $f_\alpha \downarrow 0$ we have $T(f_\alpha) \downarrow 0$.

From LDCT, $\mathbb{E}[\cdot|\Sigma]$ is order continuous.

The mapping of f to its conditional expectation has the following important properties, which give the key to our reformulation in terms of Riesz spaces:

1. $f \mapsto \mathbb{E}[f|\Sigma]$ is linear;
2. if $f \geq 0$ then $\mathbb{E}[f|\Sigma] \geq 0$ (is positive);
3. $\mathbb{E}[\mathbf{1}|\Sigma] = \mathbf{1}$ (preserves the 1 function);
4. $\mathbb{E}[\mathbb{E}[f|\Sigma]|\Sigma] = \mathbb{E}[f|\Sigma]$ (is idempotent);
5. if $f_n \uparrow f$ in $L^1(\Omega, \mathcal{F}, P)$ then $\mathbb{E}[f_n|\Sigma] \uparrow \mathbb{E}[f|\Sigma]$ in $L^1(\Omega, \Sigma, P)$ (is order continuous).

Together 1 and 4 give that $f \mapsto \mathbb{E}[f|\Sigma]$ is a projection. To make use of 2, 3 and 5 we develop the basics of Riesz space theory.

Probability spaces

- Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, i.e. Ω is a set, \mathcal{F} is a σ -algebra of subsets of Ω and μ is a positive measure on \mathcal{F} with $\mu(\Omega) = 1$.
- A random variable is an element of real valued $L^1(\Omega, \mathcal{F}, \mu)$.
- The expectation of the random variable f is

$$\mathbb{E}[f] = \int_{\Omega} f d\mu.$$

- Here $\mathbb{E}[f]$ can also be considered as the class of measurable functions on Ω with value a.e. $\int_{\Omega} f d\mu$ making

$$\mathbb{E} : L^1(\Omega, \mathcal{F}, \mu) \rightarrow L^1(\Omega, \mathcal{N}, \mu) \subset L^1(\Omega, \mathcal{F}, \mu),$$

where \mathcal{N} consists of the sets of measure zero and their complements.

Conditional expectations

- For Σ a sub- σ -algebra of \mathcal{F} , the conditional expectation of f with respect to Σ , denoted $\mathbb{E}[f|\Sigma]$, is F where F is Σ -measurable and for each $A \in \Sigma$

$$\int_A F d\mu = \int_A f d\mu. \quad (1)$$

- We observe that

$$\mathbb{E}[\cdot|\mathcal{F}] : L^1(\Omega, \mathcal{F}, \mu) \rightarrow L^1(\Omega, \Sigma, \mu) \subset L^1(\Omega, \mathcal{F}, \mu)$$

is a positive order continuous projection onto $L^1(\Omega, \Sigma, \mu)$, which maps a.e. constant functions to a.e. constant functions.

The following result relates contractive projections to conditional expectations.

- Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $1 \leq p < \infty$.
- If $T : L^p(\Omega, \mathcal{F}, \mu) \rightarrow L^p(\Omega, \mathcal{F}, \mu)$ is a positive contractive projection with $T\mathbf{1} = \mathbf{1}$.
- Then $Tf = \mathbb{E}[f \mid \Sigma]$, $f \in L^p(\Omega, \mathcal{F}, \mu)$, for a unique σ -algebra $\Sigma \subset \mathcal{F}$.
- The precise version quoted here is from Rao and attributed to Sidak.

- In the works of Douglas ¹ and Andô ² the range spaces of contractive projections on L_p spaces over probability measures are characterized as conditional expectation operators.
- In particular Douglas characterizes the ranges spaces of conditional expectation operators as the range spaces of contractive projections on L_p that are closed under order limits and are vector sub-lattices of L_p .
- The work of Douglas will be returned to later in the Riesz space setting.

¹R.G. DOUGLAS, Contractive projections on an L_1 space, *Pacific J. Math.*, **15** (1965), 443-462.

²T. ANDÔ, Contractive projections in L_p spaces, *Pacific J. Math.*, **17** (1966), 391-405.

Riesz space conditional expectation operators

We³ introduced the following definition for a conditional expectation operator in a Riesz space.

Definition

Let E be a Dedekind complete Riesz space with weak order unit. A positive order continuous projection T on E with range, $R(T)$, an order closed Riesz subspace of E , is called a conditional expectation if $T(e)$ is a weak order unit of E for each weak order unit e in E .

One can assume that T is strictly positive (T is a positive operator with $Tf \neq 0$ for $f \in E_+ \setminus \{0\}$), by working in a quotient space if necessary.

³W.-C. KUO, C.C.A. LABUSCHAGNE, B.A. WATSON, Discrete time stochastic processes on Riesz spaces, *Indag. Mathem.*, **15**(2004), 435-451.

- If $E = L^1(\Omega, \mathcal{F}, \mu)$ is a probability space and Σ is a sub- σ -algebra of \mathcal{F} , then E is a Dedekind complete Riesz space with weak order unit $e = \mathbf{1}$ and

$$Tf = \mathbb{E}[f|\Sigma]$$

is a Riesz space conditional expectation operator on E with $Te = e$.

- The converse is also true. If T is a Riesz space strictly positive conditional expectation operator on $E = L^1(\Omega, \mathcal{F}, \mu)$, where μ is a probability measure, then there is a sub- σ -algebra, Σ , of \mathcal{F} so that $T = \mathbb{E}[\cdot|\Sigma]$.

Example - σ -finite processes

- Dellacherie and Meyer considered σ -finite processes in the context of martingale theory.⁴
- Let $(\Omega, \mathcal{A}, \mu)$ be a σ -finite measure space and let $(\Omega_i)_{i \in \mathbb{N}}$ be a μ -measurable partition of Ω into sets of finite positive measure.
- Let \mathcal{A}_0 be the sub- σ -algebra of \mathcal{A} generated by $(\Omega_i)_{i \in \mathbb{N}}$. We take the Riesz space

$$E = \{f \in L^0(\Omega, \mathcal{A}, \mu) \mid f|_{\Omega_i} \in L^\infty(\Omega_i, \mathcal{A}_i, \mu|_{\mathcal{A}_i})$$

where $\mathcal{A}_i = \{U \cap \Omega_i \mid U \in \mathcal{A}\}$.

- Here E is a Riesz space under a.e. operations.

⁴Sections 39, 42 and 43 of C. DELLACHERIE, P.-A. MEYER, *Probabilities and Potentials: B, Theory of Martingales*, North Holland Publishing Company, 1982.

Example - σ -finite processes

- For $f \in E$ define

$$Tf(\omega) := \frac{\int_{\Omega_i} f d\mu}{\mu(\Omega_i)}, \quad \text{for } \omega \in \Omega_i. \quad (2)$$

- Then T is a Riesz space conditional expectation operator on E .
- For $f \in E \cap L^1(\Omega, \mathcal{A}, \mu)$, we have $Tf = \mathbb{E}[f|\mathcal{A}_0]$.
- But we have neither $L^1(\Omega, \mathcal{A}, \mu) \subset E$ nor $E \subset L^1(\Omega, \mathcal{A}, \mu)$.

T -universal extension of E

- Each Riesz space E with weak order unit and conditional expectation operator T admits a T -universal completion (the maximal Riesz space to which T can be extended as a conditional expectation operator).
- The T -universal completion of E is

$$\hat{E} = \text{dom}(T) - \text{dom}(T)$$

where

$$\text{dom}(T) := \{f \in E_+^u \mid \exists \text{ net } f_\alpha \uparrow f \text{ in } E^u, (f_\alpha) \subset E_+, Tf_\alpha \text{ bdd in } E^u\}.$$

- Here E^u is the universal completion of E and E_+^u is the positive cone of E^u .
- Here \hat{E} is a Dedekind complete Riesz space containing E as an order dense subspace, and having weak order unit.
- The space \hat{E} will be denoted $L^1(T)$.

Extension of T to \hat{E}

- Each Riesz space conditional expectation operator admits unique extension to a conditional expectation on the T -universal completion, \hat{E} , of E .
- If $f \in \hat{E}_+$ then there is a net (f_α) in E_+ with (Tf_α) bounded in the universal completion, E^u , of E , with $f_\alpha \uparrow f$.
- Then $Tf_\alpha \uparrow$ in E and as such has limit, which we denote $\hat{T}(f)$, in \hat{E} .
- We extend \hat{T} to the whole of \hat{E} by setting

$$\hat{T}f = \hat{T}f^+ - \hat{T}f^-, \quad f \in \hat{E}.$$

- It can now be verified that the extension $\hat{T} : \hat{E} \rightarrow \hat{E}$ of T is a conditional expectation operator on \hat{E} .

Example - σ -finite processes

- Here

$$\hat{E} = \left\{ f \in L^0(\Omega, \mathcal{A}, \mu) \mid \int_{\Omega_i} |f| d\mu < \infty \text{ for all } i \in \mathbb{N} \right\},$$

which is characterized by $f|_{\Omega_i} \in L^1(\Omega, \mathcal{A}, \mu)$, for each $i \in \mathbb{N}$.

-

$$\hat{T}f(\omega) := \frac{\int_{\Omega_i} f d\mu}{\mu(\Omega_i)}, \quad \text{for } \omega \in \Omega_i. \quad (3)$$

- E has weak order unit $e = 1$, the function identically 1 on Ω , which again is a weak order unit for E , but is not in general in $L^1(\Omega, \mathcal{A}, \mu)$.
- The range of the generalized conditional expectation operator T is

$$R(T) = \{f \in L^0(\Omega, \mathcal{A}, \mu) \mid f \text{ a.e. constant on } \Omega_i, i \in \mathbb{N}\},$$

which is an algebra.

Commutation in $L^1(\Omega, \mathcal{A}, \mu)$

- For Σ a sub- σ -algebra of \mathcal{F} , the conditional expectation of f with respect to Σ , denoted $\mathbb{E}[f|\Sigma]$, is defined to be in $L^1(\Omega, \Sigma, \mu)$ with



$$\int_{\Omega} \chi_A \mathbb{E}[f|\Sigma] d\mu = \int_{\Omega} \chi_A f d\mu$$

for each $A \in \Sigma$. So by the uniqueness of the conditional expectation, $\mathbb{E}[\chi_A f|\Sigma] = \chi_A \mathbb{E}[f|\Sigma]$.

Commutation in Riesz spaces

- Let E be a Dedekind complete Riesz space with weak order unit, T a conditional expectation on E and B the band in E generated by $g \in R(T) \cap C_e$, with associated band projection P_g . Then $TP_g = P_gT$.
- Why?
- Let $f \in E_+$ then

$$P_g f = \sup_{n \in \mathbb{N}} (ng) \wedge f = o - \lim_{n \rightarrow \infty} (ng) \wedge f.$$

- But T is order continuous so

$$TP_g f = o - \lim_{n \rightarrow \infty} T[(ng) \wedge f].$$

- But as T is a positive operator

$$T[(ng) \wedge f] \leq (nTg) \wedge Tf = (ng) \wedge Tf.$$

- So taking order limits gives

$$TP_g f \leq P_g Tf.$$

Commutation in Riesz spaces

- Now $P_{e-g} = I - P_g$ so

$$Tf - TP_gf = TP_{e-g}f \leq P_{e-g}Tf = Tf - P_gTf.$$

- Thus $P_gTf \leq TP_gf$.
- Combining with the previous slide gives $P_gTf = TP_gf$ for $f \in E_+$.
- For $f \in E$, decomposing f in to f^\pm , we have $P_gTf^\pm = TP_gf^\pm$.
- Thus

$$P_gTf = P_gTf^+ - P_gTf^- = TP_gf^+ - TP_gf^- = TP_gf.$$

- This provides the starting point of the averaging property of conditional expectation operators.

Thank you!