$\mathbb{L}$-functional analysis

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Kaplansky-Hilbert modules

- **Extremally disconnected**: closure of open sets is open
- **Stonean**: extremally disconnected compact Hausdorff space
- AW*-algebra: C*-algebra with some extra assumptions (slight generalization of von Neumann algebra)
- abelian AW*-algebra: $C(K)$ with K Stonean

Kaplansky, 1953: initiated study of Kaplansky-Hilbert modules (KH-modules): Hilbert spaces H with $\mathbb{C}$ replaced by an abelian AW*-algebra $\mathbb{A}$.

- H is an $\mathbb{A}$-module
- $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{A}$, positive definite, $\mathbb{A}$-sesquilinear
- Some completeness assumption

Kaplansky used KH-modules to characterize type I AW*-algebras.

Hilbert C*-modules

1970's: theory of Hilbert C*-modules; $\mathbb{A}$ is generalized to an arbitrary C*-algebra.

- Now an enormous area of research
- Very important in noncommutative geometry

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- Very important in noncommutative geometry

The theory of Hilbert C*-modules is not as nice as the theory of KH-modules:

- Only a weak form of Cauchy-Schwarz
- No Riesz Representation Theorem ($H^* \cong H$)
- $V \subseteq H$ submodule: $V^{\perp\perp} \neq \overline{V}$

The theory of Hilbert C*-modules is quite different from the theory of KH-modules

2021

Some scattered results about KH-modules appeared in the literature, until:

- Edeko, Haasse, Kreidler (2021, arxiv): A Decomposition Theorem for Unitary Group Representations on Kaplansky-Hilbert Modules and the Furstenberg-Zimmer Structure Theorem

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Quick summary of the paper:

- Elementary proof of Spectral Theorem for Hilbert-Schmidt operators on KH-modules
- Theorem on unitary group representations on KH-modules
- Combine these to obtain KH-module theoretic proof of the famous Furstenberg-Zimmer Structure Theorem
- This removes separability restrictions
- Future KH-module theory applications to ergodic theory

Probability theory in vector lattices

- Invented in 2000's in South Africa
- E (replacing L^1) is a Dedekind complete vector lattice, e (replacing $\mathbf{1}$) weak unit
- $T : E \to E$ **conditional expectation**: linear, positive, order continuous, $Te = e$, $R(T)$ Dedekind complete
- Extend T: $R(T)$ becomes universally complete
- $R(T)$ admits a very nice f-algebra multiplication
- E becomes an $R(T)$-module
- Define $\|\cdot\|_p : E \to R(T)$ by $\|f\|_p := T(|f|^p)^{\frac{1}{p}}$
- Define $L^p(T)$ as those f for which $\|f\|_p$ exists
- Kalauch, Kuo, Watson 2023: Riesz Representation Theorem for $L^2(T)$

Connection between those two theories?

- In KH-modules, scalars: $\mathbb{A} \cong C(K)$, abelian AW*-algebra
- In probability, scalars: $R(T)$, universally complete VL

Connection between those two theories?

- In KH-modules, scalars: $\mathbb{A} \cong C(K)$, abelian AW*-algebra
- In probability, scalars: $R(T)$, universally complete VL

Both are (real/complex) Dedekind complete unital f-algebras!

Our goal: unify both theories by setting up a general theory of functional analysis, replacing $\mathbb{R}$ (or $\mathbb{C}$) by a real (or complex) Dedekind complete unital f-algebra.

Notation? $\mathbb{A}$ (for algebra) or $\mathbb{L}$ (for lattice)? Best is $\mathbb{L}$.

Q: How does $\mathbb{L}$ compare with $\mathbb{A} \cong C(K)$ and $R(T)$?

- K Stonean
- $C_\infty(K) = \{f \in C(K, [-\infty, \infty]) \colon f^{-1}(\mathbb{R}) \text{ is dense}\}$
- $R(T)$ is universally complete, so isomorphic to $C_\infty(K)$ for some Stonean K
- A Dedekind complete unital f-algebra $\mathbb{L}$ is an order ideal and subalgebra of $C_\infty(K)$ containing $C(K)$, so

$$C(K) \subseteq \mathbb{L} \subseteq C_\infty(K)$$

So the KH-module theory and the theory of probability in vector lattices correspond to the extreme cases $\mathbb{L} = C(K)$ and $\mathbb{L} = C_\infty(K)$

$\mathbb{L}$-normed spaces

From now on $\mathbb{L}$ is a fixed (real or complex) Dedekind complete unital f-algebra. Elements of $\mathbb{L}$ will be denoted by λ and μ.

Definition

An $\mathbb{L}$-**normed space** $(X, \|\cdot\|)$ is an $\mathbb{L}$-module X equipped with a map $\|\cdot\| : X \to \mathbb{L}^+$ satisfying

- $\|\lambda x\| = |\lambda| \, \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$
- $\|x\| = 0 \Leftrightarrow x = 0$.

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An $\mathbb{L}$-normed space is an example of a **lattice normed space**, which goes back to Kantorovich (1936), who investigated mostly the non-module case.

Example

$(\mathbb{L}, |\cdot|)$ is an $\mathbb{L}$-normed space

Convergence

We define $x_\alpha \to x$ to mean that $\|x_\alpha - x\| \to 0$ in $\mathbb{L}$, so we need a notion of convergence in $\mathbb{L}$. **Order convergence** is used in both motivating examples.

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Definition

Let X be an $\mathbb{L}$-normed space, (x_α) a net in X, and $x \in X$. Then we define $x_\alpha \to x$ to mean that

$$\exists \mathcal{E} \searrow 0 \; \forall \varepsilon \in \mathcal{E} \; \exists \alpha_0 \; \forall \alpha \geq \alpha_0 \;\; \|x_\alpha - x\| \leq \varepsilon.$$

Similar to convergence in $\mathbb{R}$, except $\mathcal{E}$ depends on the net (x_α). Note that notion of convergence in X is **not** topological! It turns X into a **convergence space**.

Completeness

Definition

A net (x_α) in an L-normed space X is **Cauchy** if $(x_\alpha - x_\beta) \to 0$. X is **complete** or an $\mathbb{L}$-**Banach space** if every Cauchy net converges.

The Dedekind completeness of $\mathbb{L}$ is equivalent to the completeness of $(\mathbb{L}, |\cdot|)$.

Thus the Dedekind completeness assumption on $\mathbb{L}$ is necessary.

$\ell_\infty(S, \mathbb{L})$

Let S be a nonempty set.

Example

$$\ell_\infty(S, \mathbb{L}) := \{f \colon S \to \mathbb{L} \colon \exists M \in \mathbb{L}^+ \, \forall s \in S \, |f(s)| \leq M\}$$

Defining $(\lambda f)(s) := \lambda f(s)$ turns $\ell_\infty(S, \mathbb{L})$ into an $\mathbb{L}$-module, and for $f \in \ell_\infty(S, \mathbb{L})$, define (using Dedekind completeness of $\mathbb{L}$)

$$\|f\|_\infty := \sup_{s \in S} |f(s)|.$$

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Theorem

$\ell_\infty(S, \mathbb{L})$ is an $\mathbb{L}$-Banach space.

Proof is **very** similar to the classical case.

Denote $\ell_\infty^{\mathbb{L}} := \ell_\infty(\mathbb{N}, \mathbb{L})$.

Definition

We define $c_0^{\mathbb{L}}$ to be the elements $f \in \ell_\infty^{\mathbb{L}}$ such that

$$\exists \mathcal{E} \searrow 0 \; \forall \varepsilon \in \mathcal{E} \; \exists N \in \mathbb{N} \; \forall n \geq N \; |f(n)| \leq \varepsilon$$

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Theorem

$c_0^{\mathbb{L}}$ is a closed subspace of $\ell_\infty^{\mathbb{L}}$ (and hence an $\mathbb{L}$-Banach space)

Closed means that the limit of every converging net is in the set. In the classical case this is a simple 2ε-proof.

$c_0^{\mathbb{L}}$ is closed

Proof.

Let $c_0^{\mathbb{L}} \ni f_\alpha \to f \in \ell_\infty^{\mathbb{L}}$. To show: $f \in c_0^{\mathbb{L}}$. We have:

$$\forall \alpha \; \exists \mathcal{E}_\alpha \searrow 0 \; \forall \varepsilon \in \mathcal{E}_\alpha \; \exists N \in \mathbb{N} \; \forall n \geq N \; |f_\alpha(n)| \leq \varepsilon;$$

$$\exists H \searrow 0 \; \forall \eta \in H \; \exists \alpha_\eta \; \forall \alpha \geq \alpha_\eta \; \|f - f_\alpha\|_\infty \leq \eta.$$

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$$\text{Define} \quad \mathcal{E} := \{\eta + \varepsilon \colon \eta \in H, \; \varepsilon \in \mathcal{E}_{\alpha_\eta}\} = \bigcup_{\eta \in H} \left(\eta + \mathcal{E}_{\alpha_\eta}\right),$$

$$\text{then} \quad \inf \mathcal{E} = \inf_{\eta \in H} \left[\inf \left(\eta + \mathcal{E}_{\alpha_\eta}\right)\right] = \inf_{\eta \in H} \eta = 0.$$

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Let $\eta + \varepsilon \in \mathcal{E}$ ($\eta \in H$, $\varepsilon \in \mathcal{E}_{\alpha_\eta}$) be arbitrary. Let $N \in \mathbb{N}$ be such that $|f_{\alpha_\eta}(n)| \leq \varepsilon$ for all $n \geq N$. Then for $n \geq N$:

$$|f(n)| \leq |f(n) - f_{\alpha_\eta}(n)| + |f_{\alpha_\eta}(n)| \leq \eta + \varepsilon.$$

Operators

- X, Y $\mathbb{L}$-normed spaces, $T \in \mathrm{Hom}_{\mathbb{L}}(X, Y)$.

Definition

T is **bounded** if $\exists M \in \mathbb{L}^+ \ \forall x \in X \ \ \|Tx\|_Y \leq M \|x\|_X$.

$$\|T\| := \inf\{M \in \mathbb{L}^+ : \forall x \in X \ \ \|Tx\| \leq M \|x\|\}$$

$$B(X, Y) := \{T \in \mathrm{Hom}_{\mathbb{L}}(X, Y) : \ T \text{ is bounded}\}.$$

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Theorem

$B(X, Y)$ is an $\mathbb{L}$-normed space satisfying $\|TS\| \leq \|T\| \, \|S\|$ which is complete whenever Y is complete.

Proof is very similar to the classical case.

$\mathbb{L}$-Banach algebras

Definition

A is an $\mathbb{L}$-**Banach algebra** if it is an $\mathbb{L}$-Banach space equipped with a 'bilinear' map $A \times A \to A$ satisfying $\|ab\| \leq \|a\| \, \|b\|$.

The sentence 'Let A be an $\mathbb{A}$-Banach algebra' would be awkward, hence $\mathbb{L}$ is a better notation than $\mathbb{A}$.

Example

If X is an $\mathbb{L}$-Banach space, then $B(X)$ is an $\mathbb{L}$-Banach algebra.

$\varphi\colon X \to \mathbb{L}$ is **sublinear** if $\varphi(\lambda x) = \lambda\varphi(x)$ and $\varphi(x+y) \leq \varphi(x) + \varphi(y)$ for $\lambda \in \mathbb{L}^+$ and $x, y \in X$.

Theorem

Let X be a real $\mathbb{L}$-module, $Y \subseteq X$ submodule, $f \in \mathrm{Hom}_{\mathbb{L}}(Y, \mathbb{L})$, $\varphi\colon X \to \mathbb{L}$ sublinear with $f(y) \leq \varphi(y)$ for all $y \in Y$. Then there exists an $F \in \mathrm{hom}_{\mathbb{L}}(X, \mathbb{L})$ extending f with $F(x) \leq \varphi(x)$ for all $x \in X$.

Classical proof relies on the fact that if $\lambda \neq 0$, then (λ is invertible) and ($\lambda > 0$ or $\lambda < 0$). Neither hold in $\mathbb{L}$ so the proof is a lot more sophisticated.

- $X^* := B(X, \mathbb{L})$
- 1 is the unit in $\mathbb{L}$

Corollary

Let X be an $\mathbb{L}$-normed space and $x \in X$, then there exists an $x^* \in X^*$ with $x^*(x) = \|x\|$ and $\|x^*\| \leq 1$.

Corollary

$J\colon X \to X^{**}$ is isometric.

Corollary

The completion of X can be defined as $\overline{J(X)}$ in X^{**}.

This circumvents set-theoretic issues with having to consider equivalence classes of Cauchy nets.

$\mathbb{L}$-Hilbert spaces

Let H be an $\mathbb{L}$-module.

Definition

An **inner product** is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{L}$ satisfying

- $\langle x, x \rangle \in \mathbb{L}^+$, and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
- $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$

Note that $\|x\| := \sqrt{\langle x, x \rangle}$ turns H into an $\mathbb{L}$-normed space; if it is complete, H is called an $\mathbb{L}$-**Hilbert space**.

Theorem (Cauchy-Schwarz, Pythagoras)

$|\langle x, y \rangle| \leq \|x\| \|y\| \qquad \langle x, y \rangle = 0 \Rightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2.$

The next results requires more work.

Theorem (Parallelogram law)

Let X be an $\mathbb{L}$-normed space. Then $\|\cdot\|$ is derived from an inner product if and only if for all $x, y \in X$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

C is $\mathbb{L}$-**convex** if $\lambda x + (1-\lambda)y \in C$ for any $0 \leq \lambda \leq 1$ and $x, y \in C$.

Theorem

Let H be an $\mathbb{L}$-Hilbert space and K be a closed, $\mathbb{L}$-convex nonempty subset of H. Then, for any $x \in H$, there exists a unique point k_0 in K such that

$$\|x - k_0\| = \inf_{k \in K} \|x - k\| .$$

By adapting Kaplansky's proof from 1953, we also obtain

Theorem (Riesz Representation Theorem)

Let H be an $\mathbb{L}$-Hilbert space and $f \in H^*$. Then there exists a unique $y \in H$ with $f(x) = \langle x, y \rangle$ for all $x \in H$.

Corollary

Every $T \in B(H)$ has an adjoint.

Definition

A is an $\mathbb{L}$-**C*-algebra** if it is a $\mathbb{L}$-Banach *-algebra with $\|a^*\| = \|a\|$ and $\|a^* a\| = \|a\|^2$.

Example

If H is an $\mathbb{L}$-Hilbert space, then $B(H)$ is an $\mathbb{L}$-C*-algebra.

Conclusion

$\mathbb{L}$-functional analysis is a nice theory where most results hold just as in the classical case, sometimes needing more sophisticated arguments.

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$\mathbb{L}$-functional analysis is a nice theory where most results hold just as in the classical case, sometimes needing more sophisticated arguments.

BUT

At the moment it is 'just' a unifying theory. It would be nice to find an application.

Future work:

- Generalize the rest of functional analysis to $\mathbb{L}$-functional analysis;
- Find a nice application.

Thank you for your attention!