Nonlinear Perron-Frobenius Theory: Part 2

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Hilbert's metric on cones

Hilbert's (projective) metric on C is given by,

$$d_H(x, y) = \log\left(\frac{M(x/y)}{m(x/y)}\right) \quad \text{for all } x \sim_C y.$$

where

$$M(x/y) = \inf\{\beta > 0 \colon x \leq \beta y\}$$

and

$$m(x/y) = \sup\{\alpha > 0 \colon \alpha y \leq x\}.$$

(Pioneered by Garrett Birkhoff and Hans Samelson)

Note that

$$d_H(\lambda x, \mu y) = d_H(x, y) \text{ for all } \lambda, \mu > 0 \text{ and } x \sim_C y.$$

Hilbert's metric is a genuine metric on the set of rays in each part of C, if C is a closed cone in a normed space.

Nonexpansiveness

Suppose C and K are cones in V and W, resp. If $h\colon C \to K$ is an order-preserving homogenous (degree 1) map, then for $x, y \in C$ with

$$\alpha y \leq x \leq \beta y,$$

we have that

$$\alpha h(y) \leq h(x) \leq \beta h(y).$$

Thus,

$$M(h(x)/h(y)) \leq M(x/y) \quad \text{and} \quad m(x/y) \leq m(h(x)/h(y))$$

and hence h is nonexpansive under d_H, i.e.,

$$d_H(h(x), h(y)) \leq d_H(x, y) \quad \text{for all } x \sim_C y$$

Birkhoff's contraction ratio

Given a linear map $L\colon V \to W$ with $L(C) \subseteq K$ the **Birkhoff contraction ratio** is defined by

$$\kappa(L) = \inf\{c \geq 0\colon d_H(Lx, Ly) \leq c d_H(x, y) \text{ for all } x \sim_C y\}.$$

When is $\kappa(L) < 1$ and how can we compute it?

The **projective diameter** of L is given by

$$\Delta(L) = \sup\{d_H(Lx, Ly)\colon x, y \in C \text{ with } Lx \sim_K Ly\}.$$

Birkhoff's theorem

Theorem Let C be a cone in a vector space V and K be a cone in a vector space W. If $L\colon V \to W$ is a linear map with $L(C) \subseteq K$, then

$$\kappa(L) = \tanh\Big(\frac{1}{4}\Delta(L)\Big),$$

where $\tanh(\infty) = \frac{e^{\infty}-e^{-\infty}}{e^{\infty}+e^{-\infty}} = 1$.

So, if $\Delta(L) < \infty$, then L is a Lipschitz contraction on each part of C, with contraction constant $\tanh(\Delta(L)/4) < 1$.

Bauer, Bushell, Eveson, Hopf, Krasnoselskii, Nussbaum, Ostrowski, Thompson, ...

Banach fixed point theorem

If (V, C, u) is a complete order-unit space and $M = \{x \in C^\circ \colon \|x\|_u = 1\}$, then the topology of the metric space (M, d_H) coincides with the norm topology, so it is complete.

If $L \colon V \to V$ is a linear map with $L(C^\circ) \subseteq C^\circ$ and $\Delta(L) < \infty$, then

$$x \in M \mapsto \frac{Lx}{\|Lx\|} \in M$$

is a contraction mapping, hence it has a unique fixed point.

This implies that L has a unique eigenvector in C°.

Positive matrices

If L is given by a positive $m \times n$ matrices $A = (a_{ij})$, so $a_{ij} > 0$ for all i and j, there exists the following explicit formula:

$$\Delta(A) = \max_{i,j} d_H(Ae_i, Ae_j) = \log \left(\max_{i,j,p,q} \frac{a_{pi}a_{qj}}{a_{pj}a_{qi}} \right) < \infty,$$

where $e_1, \ldots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$.

Thompson's metric

Given a cone in an order unit space (V, C, u) and $x, y \in C^\circ$ one defines

$$d_T(x, y) = \max\{\log M(x/y), \log M(y/x)\}.$$

It is a **metric on** C° and order-preserving homogeneous maps are d_T-**nonexpansive**.

For the cone $(\mathbb{R}^n_+)^\circ$ we have that

$$M(x/y) = \max_i \frac{x_i}{y_i},$$

So,

$$d_T(x, y) = \max_i |\log x_i - \log y_i| = \|\log x - \log y\|_\infty$$

Hence the map $\log\colon ((\mathbb{R}^n_+)^\circ, d_T) \to (\mathbb{R}^n, \|\cdot\|_\infty)$ is an **isometry**.

Log-Exp Transform

Given a topical map $f\colon \mathbb{R}^n \to \mathbb{R}^n$, the coordinate-wise exp and log functions yield commutative diagram:

$$
\begin{array}{ccccc}
 & (\mathbb{R}^n_+)^\circ & \xrightarrow{h} & (\mathbb{R}^n_+)^\circ & \\
\log & \downarrow & & \downarrow & \log \\
 & \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^n &
\end{array}
$$

The induced map $h\colon (\mathbb{R}^n_+)^\circ \to (\mathbb{R}^n_+)^\circ$ is order-preserving with respect to $\mathbb{R}^n_+$ and **homogeneous (of degree 1)**, so

$$h(\lambda x) = \lambda h(x) \quad \text{for all } x \in (\mathbb{R}^n_+)^\circ \text{ and } \lambda \geq 0.$$

Topical maps are nonexpansive under the $\|\cdot\|_\infty$-norm

Cone spectral radius

Let K be a closed cone in a Banach space.

For a continuous order-preserving homogeneous map $f\colon K \to K$ the **(Bonsall) cone spectral radius** is defined by

$$r_K(f) = \lim_{m\to\infty} \|f^m\|_K^{1/m} = \inf_{m\geq 1} \|f^m\|_K^{1/m},$$

where $\|f\|_K = \sup\{\|f(x)\|\colon x \in K \text{ and } \|x\| \leq 1\}$.

Theorem (Nussbaum) If $f\colon K \to K$ is a compact, continuous, homogeneous order-preserving map and $r_K(f) > 0$, then there exists $v \in K \setminus \{0\}$ with

$$f(v) = r_K(f)v.$$

Cone spectrum

Given a homogeneous order-preserving map $f\colon K \to K$ the **cone spectrum** is given by

$$\sigma_K(f) = \{\lambda \geq 0\colon f(v) = \lambda v \text{ for some } v \neq 0 \text{ in } K\}$$

Theorem (L&Nussbaum) Let K be a **polyhedral** cone with m faces. If $f\colon K \to K$ is a homogeneous order-preserving map, then $|\sigma_K(f)| \leq m - 1$ and this bound is sharp.

Theorem (L&Nussbaum) If K is a **non-polyhedral** cone, then there exists a continuous homogeneous order-preserving map $f\colon K \to K$ with **infinitely** many distinct eigenvalues.

Example

If

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \partial\Pi_2(\mathbb{R})$$

and $f\colon \Pi_2(\mathbb{R}) \to \Pi_2(\mathbb{R})$ is given by

$$f(X) = \Big(\mathrm{tr}(XA)X\Big)^{1/2} \quad \text{for } X \in \Pi_2(\mathbb{R}).$$

Then

$$\sigma_{\Pi_2(\mathbb{R})}(f) = [0, 1].$$

Continuity cone spectral radius

Open problem Given a continuous homogeneous order-preserving map $f \colon K \to K$, when is true that for each sequence $(f_k)_k$ of continuous homogeneous order-preserving maps on K with

$$\lim_{k\to\infty} \|f - f_k\|_K = 0,$$

we have that $\lim_{k\to\infty} r_K(f_k) = r_K(f)$?

Theorem (L&Nussbaum) If $f \colon K \to K$ is a compact, continuous, homogeneous order-preserving map and either, $r_K(f) = 0$, or, there exist $0 < a_1 < a_2 < a_3 < \ldots$ **not** in $\sigma_K(f)$ with

$$\lim_{k\to\infty} a_k = r_K(f),$$

then f has a continuous cone spectral radius.

Eigenvectors in the interior

Problem When does a homogeneous order-preserving map $f\colon K^\circ \to K^\circ$ have an eigenvector $v \in K^\circ$?

A topical map $g\colon \mathbb{R}^n \to \mathbb{R}^n$ has an **additive eigenvector** $z \in \mathbb{R}^n$, i.e.,

$$g(z) = \mu\mathbf{1} + z \quad \text{for some } \mu \in \mathbb{R},$$

if and only if the log-exp transform $f\colon (\mathbb{R}^n_+)^\circ \to (\mathbb{R}^n_+)^\circ$ has an eigenvector $v \in (\mathbb{R}^n_+)^\circ$ with $f(v) = e^\mu v$.

Difficult problem and there seems to be no clear-cut solution.
Akian, Gaubert, Gunawardena, Hochart, Krause, Lemmens, Lins, Nussbaum, Schneider, Turner, Vigeral, ...

Perturbation approach

A **contractive perturbation**, $f_\varepsilon \colon K^\circ \to K^\circ$ given by,

$$f_\varepsilon(x) = f(x) + \varepsilon\vartheta(x)w \quad \text{for } x \in K^\circ,$$

where $\varepsilon > 0$, $\vartheta \in (K^*)^\circ$ and $w \in K^\circ$.

Note f_ε is **contractive under Hilbert's metric**.

Lemma Let $K \subseteq V$ be a solid closed cone and $\varphi \in (K^*)^\circ$. If $f \colon K^\circ \to K^\circ$ is a homogeneous order-preserving map and f_ε is a contractive perturbation of f, then for each $\varepsilon > 0$, f_ε has a **unique eigenvector** $x^\varepsilon \in K^\circ$ with $\varphi(x^\varepsilon) = 1$ and

$$\lim_{k\to\infty} \frac{f_\varepsilon^k(x)}{\varphi(f_\varepsilon^k(x))} = x^\varepsilon \quad \text{for all } x \in K^\circ.$$

Digraphs 1

Theorem If $f\colon K^\circ \to K^\circ$ is a homogeneous order-preserving map and $f_\varepsilon\colon K^\circ \to K^\circ$ is a contractive perturbation of f with eigenvector $x^\varepsilon \in K^\circ$ and $\varphi(x^\varepsilon) = 1$, then f has an eigenvector $v \in K^\circ$ if and only if $\mathrm{cl}(\{x^\varepsilon \in K^\circ\colon \varepsilon > 0\})$ is **compact** in K°.

For $j \in \{1, \dots, n\}$ and a positive real $u > 0$ define $u^j \in \mathbb{R}^n_+$ by

$$u^j_i = u \text{ if } i = j, \text{ and } u^j_i = 1 \text{ otherwise.}$$

We associate a **digraph** $G_f = (V, A)$ to a homogeneous order-preserving map $f\colon (\mathbb{R}^n_+)^\circ \to (\mathbb{R}^n_+)^\circ$ as follows: the vertex set $V = \{1, \dots, n\}$ and there exists an arc $(i, j) \in A$ if

$$\lim_{u \to \infty} f_i(u^j) = \infty.$$

Digraphs 2

Theorem (Gaubert-Gunawardena) Let $f\colon (\mathbb{R}^n_+)^\circ \to (\mathbb{R}^n_+)^\circ$ is a homogeneous order-preserving map and denote $\Delta_n^\circ = \{x \in (\mathbb{R}^n_+)^\circ \colon \sum_i x_i = 1\}$. If G_f is **strongly connected**, then the following assertions hold:

(i) There exists an eigenvector of f in $(\mathbb{R}^n_+)^\circ$.

(ii) If f_ε is a contractive perturbation of f with eigenvector $x^\varepsilon \in \Delta_n^\circ$ for $\varepsilon > 0$, then $\mathrm{cl}(\{x^\varepsilon \in \Delta_n^\circ \colon \varepsilon > 0\})$ is a compact subset of Δ_n°.

(iii) The set $\{v \in \Delta_n^\circ \colon f(v) = r_{\mathbb{R}^n_+}(f)v\}$ is a compact subset of Δ_n°.

Dynamics

Given a continuous homogeneous map $f\colon K \to K$ it is interesting to understand the long-term behaviour of the iterates,

$$x, f(x), f^2(x) = f(f(x)), f^3(x) = f(f(f(x))), \ldots$$

Theorem (L&Scheutzow) If $f\colon X \to X$, where $X \subseteq \mathbb{R}^n$ is closed. is nonexpansive under the sup-norm $\|\cdot\|_\infty$, then every bounded orbit of f converges to a periodic orbit. Moreover, the periods of periodic points of f do not exceed

$$\max_k 2^k \binom{n}{k} \leq C3^n/\sqrt{n}.$$

Conjecture (Nussbaum) The optimal upper bound is 2^n.

Blokhuis & Wilbrink, Martus, Lyons & Nussbaum, Sine, Weller

Orbits in interior

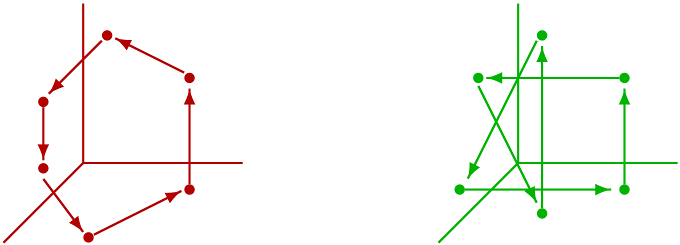
Theorem (L&Scheutzow) If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a topical map and $x \in \mathbb{R}^n$ is a periodic point of f with period p, then

$$p \leq \binom{n}{\lfloor n/2 \rfloor}$$

and this upper bound is **sharp**.

Corrolary If $K \subseteq V$ is a polyhedral cone with N facets and $f\colon K \to K$ is an order-preserving homogeneous map with $f(K^\circ) \subseteq K^\circ$, then the period of each periodic point $x \in K^\circ$ does not exceed $\binom{N}{\lfloor N/2 \rfloor}$.

Orbits in the whole polyhedral cone



Theorem (Akian, Gaubert, L& Nussbaum) If $f\colon K \to K$ is a continuous order-preserving homogeneous map on a polyhedral cone $K \subseteq V$ and $x \in K$ has a norm bounded orbit, then there exists a periodic point $\xi \in K$ of f with period p such that

$$\lim_{k\to\infty} f^{kp}(x) = \xi.$$

Periodic orbits

Theorem (Akian, Gaubert, L& Nussbaum) If $f\colon K \to K$ is an order-preserving homogeneous map on a polyhedral cone K with N facets and $x \in K$ is a periodic point of f with period p, then

$$p \leq \frac{N!}{\lfloor \frac{N}{3} \rfloor ! \lfloor \frac{N+1}{3} \rfloor ! \lfloor \frac{N+2}{3} \rfloor !}.$$

$$B(n) = \{q_1 q_2 \colon q_1, q_2 \in \mathbb{N} \text{ such that } 1 \leq q_1 \leq \binom{n}{m} \text{ and } 1 \leq q_2 \leq \binom{m}{\lfloor m/2 \rfloor} \text{ for some integer } 1 \leq m \leq n\}.$$

Theorem (L&Sparrow) The set of possible periods of periodic points of continuous order-preserving homogeneous maps on $\mathbb{R}^n_+$ is equal to $B(n)$.

Possible periods

The set of possible periods of periodic points of continuous order-preserving homogeneous maps on $\mathbb{R}^n_+$ is equal to $B(n)$.

n	Elements of $B(n)$
1	1
2	1, 2
3	1, 2, 3, 4, 6
4	1, 2, 3, 4, 5, 6, 8, 9, 10, 12
5	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 27, 30
6	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 48, 50, 51, 52, 54, 55, 56, 57, 60, 65, 66, 70, 72, 75, 78, 84, 90

Propaganda

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Nonlinear Perron-Frobenius Theory,
Cambridge Tracts Math. **189**, Cambridge Press, 2012.