A Riesz-Frechet theorem in Riesz spaces

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Abstract

Let F is a Dedekind complete Riesz space with weak order unit and conditional expectation operator T. In addition we assume that T is strictly positive and that F is T-universally complete. We define $E = L^2(T) := \{f \in F \,|\, f^2 \in F\}$, where multiplication is as defined in the f-algebra F_u, the universal completion of F. The T-strong dual of $L^2(T)$ denoted by $\hat{E}$ consists of the maps $\mathfrak{f} : L^2(T) \to R(T) := T(F)$ such that $\mathfrak{f}$ is $R(T)$-homogeneous, order continuous and there exists $k \in R(T)_+$ so that $|\mathfrak{f}(g)| \leq k\|g\|_{T,2}$ for all $g \in L^2(T)$. Here $\|g\|_{T,2} = \sqrt{T(g^2)}$ and the space $\hat{E}$ has $R(T)$ valued norm $\|\mathfrak{f}\| := \inf\{k \in R(T)_+ \,|\, |\mathfrak{f}(g)| \leq k\|g\|_{T,2}$ for all $g \in E\}$. We give a Riesz-Frechet theorem which provides an isometry between E and $\hat{E}$.

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Hahn-Jordan decomposition I

- Let E be a Dedekind complete Riesz space with weak order unit, say e, and G be a Dedekind complete Riesz subspace of E which also contains e, so $\{e\} \subset G \subset E$.
- Denote the set of all components of e in E by $\mathcal{K}(e)$, and by $\mathcal{K}_G(e) = G \cap \mathcal{K}(e)$ the set of all components of e in E which are in G.
- It should be noted here that $\mathcal{K}(e)$ and $\mathcal{K}_G(e)$ are Boolean algebras with $\mathcal{K}_G(e)$ a subalgebra of $\mathcal{K}(e)$.
- It should be noted that E is an E_e-module.
- As e is a weak order unit for G, we have that G and E are G_e-modules.

Hahn-Jordan decomposition II

- Let $\mathcal{B}$ be a Boolean subalgebra of $\mathcal{K}(e)$ which contains $\mathcal{K}_G(e)$ and is order closed in E, i.e., if (p_α) is a net in $\mathcal{B}$ with $p_\alpha \downarrow p$ in E, then $p \in \mathcal{B}$.
- Take $\psi : \mathcal{B} \to G$ to be a map such that:
  - (i) If $p \in \mathcal{B}$ and $q \in \mathcal{K}_G(e)$ then $\psi(pq) = q\psi(p)$.
  - (ii) If $p, q \in \mathcal{B}$ with $pq = 0$ then $\psi(p \vee q) = \psi(p + q) = \psi(p) + \psi(q)$ (additivity).
  - (iii) If (p_α) is a net in $\mathcal{B}$ with $p_\alpha \downarrow p$ in E, then $\psi(p_\alpha) \to \psi(p)$ (order continuity of ψ).
  - (iv) There is $g \in E^+$ so that $|\psi(p)| \leq g$ for all $p \in \mathcal{B}$.
- We say that $q \in \mathcal{B}$ is strongly positive (resp. strongly negative) with respect to ψ if $\psi(p) \geq 0$ (resp. ≤ 0) for all $p \in \mathcal{B}$ with $p \leq q$.
- By (i) with $q = 0$ we have $\psi(0) = 0$.
- The Hahn-Jordan decomposition presented gives the existence of $q \in \mathcal{B}$ so that q is strongly positive with respect to ψ and $e - q$ is strongly negative with respect to ψ.

Hahn-Jordan decomposition III

- For $q \in \mathcal{B}$ we set

$$C(q) := \{\psi(p) \,|\, p \in \mathcal{B}, p \leq q\}.$$

- Since $0 \in \mathcal{B}$, $0 \leq q$ and $\psi(0) = 0$ we have that $0 \in C(q)$. Further, as ψ is order bounded, so is $C(q)$.
- As G is Dedekind complete, we can define

$$\alpha(q) := \sup C(q) \in G^+.$$

- For $q \in \mathcal{B}$ let

$$\mathcal{M}(q) := \{p \in \mathcal{B} \mid 2\psi(p) \geq p\alpha(q), p \leq q\}.$$

- If $r, s \in \mathcal{M}(q)$ with $rs = 0$, then $r + s \in \mathcal{M}(q)$.
- For each $q \in \mathcal{B}$, $0 \in \mathcal{M}(q)$, $\mathcal{M}(q)$ has a maximal element, $\hat{q} \in \mathcal{B}$.

Hahn-Jordan decomposition IV

- For $q \in \mathcal{B}$ and each $p \in \mathcal{M}(q)$,

$$2p\psi(p) \geq p\alpha(q),$$

  so $(\alpha(q) - 2\psi(p))^+$ and p are disjoint in E.
- For $q \in \mathcal{B}$, if $\hat{q}$ is a maximal element of $\mathcal{M}(q)$, then

$$0 \leq \alpha(\hat{q}) \leq \alpha(q) \leq 2\psi(\hat{q}) \leq 2\alpha(\hat{q}). \tag{1}$$

Hahn-Jordan decomposition V

If $p \in \mathcal{B}$ with $\psi(p) < 0$ then there exists $v \in \mathcal{B}$ with $v \leq p$ and v strongly negative with respect to ψ and $\psi(v) \leq \psi(p)$.

- Let $p \in \mathcal{B}$ with $\psi(p) < 0$.
- Set $\alpha_1 := \alpha(p) \geq 0$ and $p_1 := \hat{p} \in \mathcal{M}(p)$.
- Inductively we define

$$\alpha_{n+1} := \alpha\left(p - \bigvee_{i=1}^{n} p_i\right)$$

$$p_{n+1} := \widehat{\left(p - \bigvee_{i=1}^{n} p_i\right)} \in \mathcal{M}\left(p - \bigvee_{i=1}^{n} p_i\right).$$

- Now $\alpha_n \leq 2\psi(p_n)$ for all $n \in \mathbb{N}$ and $p_i p_j = 0$ for all $i \neq j$.

Hahn-Jordan decomposition VI

- $$\bar{p} := \sum_{i=1}^{\infty} p_i = \bigvee_{i=1}^{\infty} p_i \in \mathcal{B} \leq p$$
- The additivity and order continuity of ψ give that
$$0 \leq \sum_{i=1}^{\infty} \alpha_i \leq \sum_{i=1}^{\infty} 2\psi(p_i) = 2\psi(\bar{p}). \tag{2}$$
- Let $v := p - \bar{p}$ then $v \leq p$ and $\psi(v) + \psi(\bar{p}) = \psi(p)$, so, by (2), $\psi(v) \leq \psi(p)$.
- If $q \leq v$ then $q \leq p - \sum_{i=1}^{n} p_i$ making $\psi(q) \leq \alpha_{n+1}$, for all $n \in \mathbb{N}$. This together with (2) gives that
$$n\psi(q) \leq \sum_{i=2}^{n+1} \alpha_i \leq 2\psi(\bar{p}),$$
for all $n \in \mathbb{N}$. So $\psi(q) \leq 0$. Hence v is strongly negative with respect to ψ and v is as required.

Hahn-Jordan decomposition VII

Applying Zorn's Lemma to

$$\mathcal{H} := \{p \in \mathcal{B} \mid p \text{ strongly negative w.r.t. } \psi\},$$

we have:

Theorem (Abstract Hahn-Jordan Decomposition)

There exists $q \in \mathcal{B}$ which is strongly positive with respect to ψ and which has $e - q$ strongly negative with respect to ψ.

Riesz space conditional expectation operators

- Let E be a Dedekind complete Riesz space with weak order unit.
- We recall that e is a weak order unit of E if $e \geq 0$ and

$$\sup_{n\in\mathbb{N}}((ne) \wedge f) = f$$

for each $f \in E$ with $f \geq 0$.
- A positive order continuous projection T on E with range a Dedekind complete Riesz subspace of E, is called a conditional expectation[2] if $T(e)$ is a weak order unit of E for each weak order unit e of E.
- T is said to be strictly positive if $Tf = 0$ with $f \in E_+$ implies $f = 0$.

[2] W.-C. Kuo, C.C.A. Labuschagne, B.A. Watson, Discrete time stochastic processes on Riesz spaces, *Indag. Mathem.*, **15** (2004), 435-451.

T-universal completeness

- Let E be a Dedekind complete Riesz space E with weak order unit and strictly positive conditional expectation operator T. We say that E is T-universally complete if for each upwards directed net (f_α) in E_+ with (Tf_α) order bounded in E_u we have that (f_α) is order convergent to some f in E. In this case we denote E by $L^1(T)$, see [3].
- We define

$$L^2(T) := \{f \in L^1(T) \,|\, f^2 \in L^1(T)\}$$

where the multiplication is as defined in the f-algebra E_u, see[4] and [5].

[3] W.-C. KUO, C.C.A. LABUSCHAGNE, B.A. WATSON, Conditional expectations on Riesz spaces, *J. Math. Anal. Appl.*, **303** (2005), 509-521.

[4] LABUSCHAGNE, WATSON, Discrete stochastic integration in Riesz spaces, *Positivity*, **14** (2010), 859-875.

[5] Y. AZOUZI, M. TRABELSI, L^p spaces with respect to conditional expectation on Riesz spaces, *J. Math. Anal. Appl.* **447** (2017), 798-816.

$R(T)$-Modules

Let E be a T-universally complete Riesz space, where T is a conditional expectation operator on E which has a weak order unit e with $Te = e$.

- $R(T)$ is universally complete and hence an f-algebra.[6]
- $L^p(T), p = 1, 2$, are $R(T)$-modules and E_e-modules.
- T is an averaging operator in the sense that if $f \in R(T)$ and $g \in E = L^1(T)$ then $fg \in E$ with $T(fg) = fT(g)$.
- Multiplication whenever used in the talk is that of the f-algebra E_u with multiplicative identity e.
- $E_e \subset L^p(T)$.

[6] KUO, ROGANS, WATSON, Mixing inequalities, *JMAA*, **456** (2017), 992-1004.

Let E be a Dedekind complete Riesz space with weak order unit and F an f-algebra which is also a Riesz subspace of E. If E is an F-module and $\|\cdot\| : E \to F_+$ with the following three properties, then $\|\cdot\|$ will be called an F-valued norm on E.

- $\|f\| = 0$ if and only if $f = 0$,
- $\|gf\| = |g|\,\|f\|$ for all $f \in E$ and $g \in F$,
- $\|f + h\| \leq \|f\| + \|h\|$ for all $f, h \in E$.

An $R(T)$-valued norm on $L^2(T)$ for is defined by

$$\|f\|_{T,2} = (T|f|^2)^{1/2} \in R(T)_+$$

for $f \in L^2(T)$. From Grobler[7] or Azouzi and Trabelsi[8] we have the following Riesz space Hölder's inequality. If $f \in L^2(T)$ and $g \in L^2(T)$, then $gf \in L^1(T)$ and

$$\|gf\|_{T,1} \leq \|g\|_{T,2}\|f\|_{T,2}.$$

[7] J. J. GROBLER, Jensen's and martingale inequalities in Riesz spaces, *Indag. Math., N.S.*, **25** (2014), 275-295.

[8] Y. AZOUZI, M. TRABELSI, L^p spaces with respect to conditional expectation on Riesz spaces, *J. Math. Anal. Appl.* **447** (2017), 798-816.

$E = L^2(T)$ and its duals

Let $E = L^2(T)$.

- We say that a map $\mathfrak{f} : E \to R(T)$ is a T-linear functional on E if it is additive, $R(T)$-homogeneous and order continuous.
- Since $R(T)$ is a Dedekind complete Riesz space and E is a Riesz space, a linear map from E to $R(T)$ is order bounded if and only if it is order continuous.
- We denote the space of T-linear functionals on E by E^* and call it the T-dual of E.
- We note that $E^* \subset \mathcal{L}_b(E, R(T))$, since $R(T)$-homogeneity implies real linearity.
- Further as $R(T)$ is Dedekind complete, so is $\mathcal{L}_b(E, R(T))$.[9]

[9] C.D. Aliprantis, O. Burkinshaw, *Positive operators*, Academic press, 1985.

Riesz-Kantorovich Lemma

Lemma (Riesz-Kantorovich)

The space E^* is a Riesz space with respect to the partial ordering $\mathfrak{f} \leq \mathfrak{g}$ if and only if $\mathfrak{f}(x) \leq \mathfrak{g}(x)$ for all $x \in E_+$. This partial ordering is equivalent to defining the lattice operations by

$$(\mathfrak{f} \vee \mathfrak{g})(x) := \sup\{\mathfrak{f}(y) + \mathfrak{g}(z) \,|\, y, z \in E_+, y + z = x\}$$

and

$$(\mathfrak{f} \wedge \mathfrak{g})(x) := \inf\{\mathfrak{f}(y) + \mathfrak{g}(z) \,|\, y, z \in E_+, y + z = x\}$$

for all $x \in E_+$ and extending these operators to E. Here $\mathfrak{f}_\alpha \downarrow \mathfrak{o}$ in E^* if and only if $\mathfrak{f}_\alpha(x) \downarrow 0$ in E for each $x \in E_+$. E^* is Dedekind complete and an $R(T)$-module.

The strong dual

- If $\mathfrak{f} \in E^*$ and there is $k \in R(T)^+$ such that
$$|\mathfrak{f}(g)| \leq k\|g\|_{T,2}, \quad \text{for all } g \in E,$$
we say that $\mathfrak{f}$ is T-strongly bounded.
- We denote the space of T-strongly bounded T-linear functionals on E by
$$\hat{E} := \{\mathfrak{f} \in E^* \mid \mathfrak{f}\ T\text{-strongly bounded}\}$$
and refer to it as the T-strong dual of E. It is a Dedekind complete Riesz subspace of E^*.
- Further,
$$\|\mathfrak{f}\| := \inf\{k \in R(T)^+ \mid |\mathfrak{f}(g)| \leq k\|g\|_{T,2} \quad \text{for all } g \in E\}$$
defines an $R(T)$-valued norm on $\hat{E}$ with
$$|\mathfrak{f}(g)| \leq \|\mathfrak{f}\|\,\|g\|_{T,2} \tag{3}$$
for all $g \in L^2(T)$.

Riesz-Frechet Theorem I

- Let $\mathcal{B}$ denote the lattice of components of e in $L^2(T)$.
- For brevity of notation, if $f \in E^+$ then P_f will denote the band projection onto the band generated by f in E and $p_f := P_f e$ where e is the chosen weak order unit of E.
- Here p_f is a component of e in E. Further, due to E being an E_e-module, and the definition of the multiplicative structure, $P_f g = p_f g$ where on the left is the action of the band projection P_f on g and on the right is the product of p_f and g, for $g \in E$.
- For $y \in E := L^2(T)$ let $T_y(x) := T(xy)$ for all $x \in E$, then $T_y \in \hat{E}$ and

$$\|T_y\| = \|y\|_{T,2}. \tag{4}$$

  The map $\Psi : y \mapsto T_y$ is $R(T)$-homogeneous, additive and injective.

Riesz-Frechet Theorem II

- We take $G = R(T)$, $\psi(p) = \mathfrak{g}(p)$ and $\mathcal{B}$ to be the set of all components of e in E.
- The Abstract Hahn-Jordan Theorem is applicable and gives that there is $q \in \mathcal{B}$ which is strongly positive with respect to ψ and has $e - q$ strongly negative with respect to ψ. Take $q_{\mathfrak{g}}^{+} := q$.
- Hence for each $\mathfrak{g} \in \hat{E}$, there is a component $q_{\mathfrak{g}}^{+}$ of e in E so that $\mathfrak{g}(pq_{g}^{+}) \geq 0$ and $\mathfrak{g}((e - q_{g}^{+})p) \leq 0$ for all components p of e in E.

Riesz-Frechet Theorem

Theorem (Riesz-Frechet representation theorem)

For each $\mathfrak{f} \in \hat{E}$ *there exists* $y(\mathfrak{f}) \in E := L^2(T)$ *such that* $\mathfrak{f} = T_{y(\mathfrak{f})}$.

Riesz-Frechet Theorem - Proof sketch I

- Let $\mathfrak{f} \in \hat{E}$.
- We build a diadic approximation (s_n) to $y(\mathfrak{f})^+$.
- For all $p \in \mathcal{B}$ with $p \leq q^+_{\mathfrak{f}-2^{-n}kT}$ we have

$$\left(\mathfrak{f} - \frac{k}{2^n}T\right)(p) \geq 0$$

so

$$\mathfrak{f}(p) \geq 2^{-n}kT(p).$$

- For all $p \in \mathcal{B}$ with $p \leq e - q^+_{\mathfrak{f}-2^{-n}(k+1)T}$

$$\left(\mathfrak{f} - \frac{k+1}{2^n}T\right)(p) \leq 0$$

so

$$\mathfrak{f}(p) \leq 2^{-n}(k+1)T(p).$$

Riesz-Frechet Theorem - Proof sketch II

- Let
$$h_k^1 = q^+_{\mathfrak{f}-2^{-1}kT}(e - q^+_{\mathfrak{f}-2^{-1}(k+1)T})$$
then $h_k^1 h_j^1 = 0$ for all $k \neq j$ so
$$\sum_{k=0}^{\infty} h_k^1 =: q^+$$
is a component of e.
- Here $\mathfrak{f}(p) \geq 0$ for all $p \in \mathcal{B}$ with $p \leq q^+$ and $\mathfrak{f}(p) \leq 0$ for all $p \in \mathcal{B}$ with $p \leq e - q^+$.
- For $n \in \mathbb{N}$ define
$$h_{2k}^{n+1} = h_k^n q^+_{\mathfrak{f}-2^{-n-1}(2k+1)T}$$
$$h_{2k+1}^{n+1} = h_k^n(e - q^+_{\mathfrak{f}-2^{-n-1}(2k+1)T}).$$
- Here $h_k^n h_j^n = 0$ for all $k \neq j$ and
$$\sum_{k=0}^{\infty} h_k^n = q^+.$$

- The summation
$$s_n := \sum_{k=0}^{\infty} \frac{k}{2^n} h_k^n$$
gives a diadic approximation to $y(\mathfrak{f})^+$.
- For all $p \in \mathcal{B}$,
$$\frac{k}{2^n} T(ph_k^n) \leq \mathfrak{f}(ph_k^n) \leq \frac{k+1}{2^n} Tph_k^n. \tag{5}$$
- $Ts_n \leq \mathfrak{f}(q^+)$ for all $n \in \mathbb{N}$.
- (s_n) is increasing and (Ts_n) is bounded in E_u by $\mathfrak{f}(q^+)$, so (s_n) converges in $L^1(T)$ to s.
- It can be shown that s is also in $L^2(T)$.

Riesz-Frechet Theorem - Proof sketch I

- Working from (5), we have

$$T(ps_n) \leq \mathfrak{f}(pq^+) \leq T(ps_n) + \frac{1}{2^n}T(p), \tag{6}$$

Taking the order limit as $n \to \infty$ in (6) gives

$$T(ps) = \mathfrak{f}(pq^+). \tag{7}$$

- Applying Freudenthal's theorem along with the order continuity and linearity of T and $\mathfrak{f}$ to (7) we have that

$$T(gs) = \mathfrak{f}(gq^+), \tag{8}$$

for all $g \in E^+$. This extends by linearity to all $g \in E$. We note here that s is in the band generated by q^+.
- Applying the above to $-\mathfrak{f}$ gives $y(\mathfrak{f})^-$.

Riesz-Frechet Theorem - Final version

Theorem

The map Ψ *defined by* $\Psi(f)(g) := T_f(g) = T(fg)$ *for* $f, g \in L^2(T)$ *is a bijection between* $E = L^2(T)$ *and, its* $R(T)$*-homogeneous strong dual,* $\hat{E}$*. This map is additive,* $R(T)$*-homogeneous and* $R(T)$*-valued norm preserving in the sense that* $\|T_f\| = \|f\|_{T,2}$ *for all* $f \in L^2(T)$*.*

Thank you!