RGOSA

Research Group on Ordered Structures with Applications

On Polynomial Conjectures of Nilpotent Lie Groups Unitary Representations

Ali BAKLOUTI

Faculty of Sciences of Sfax
Department of Mathematics
University of Sfax, Tunisia

March 23, 2023

Université de Sfax
TUNISIE

- Let G be a connected, simply connected, nilpotent Lie group of Lie algebra $\mathfrak{g}$.
- $\exp : \mathfrak{g} \to G$ is a (bipolynomial) diffeomorphism.
- Let $\widehat{G}$ be the unitary dual of G.

Kirillov Theory for nilpotent Lie groups

- Let G be a connected, simply connected, nilpotent Lie group of Lie algebra $\mathfrak{g}$.
- $\exp : \mathfrak{g} \to G$ is a (bipolynomial) diffeomorphism.
- Let $\widehat{G}$ be the unitary dual of G.

- Let $H = \exp \mathfrak{h}$ be a closed connected subgroup of G.

- Let χ be a unitary character of H. There exists a linear form f on $\mathfrak{g}$ which vanishes on $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ such that $\chi = \chi_f$ where

$$\chi_f(\exp X) = e^{if(X)} \quad (i = \sqrt{-1}, \quad X \in \mathfrak{h}).$$

- Let $\tau(f, H) = \mathrm{Ind}_H^G \chi$ be the induced (monomial) representation of G.

Université de Sfax
TUNISIE

- Let $H = \exp \mathfrak{h}$ be a closed connected subgroup of G.
- Let χ be a unitary character of H. There exists a linear form f on $\mathfrak{g}$ which vanishes on $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ such that $\chi = \chi_f$ where

$$\chi_f(\exp X) = e^{if(X)} \ (i = \sqrt{-1}, \ X \in \mathfrak{h}).$$

- Let $\tau(f, H) = \mathrm{Ind}_H^G \chi$ be the induced (monomial) representation of G.

Université de Sfax
TUNISIE

Kirillov Theory for nilpotent Lie groups

- Let $H = \exp \mathfrak{h}$ be a closed connected subgroup of G.

- Let χ be a unitary character of H. There exists a linear form f on $\mathfrak{g}$ which vanishes on $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ such that $\chi = \chi_f$ where

$$\chi_f(\exp X) = e^{if(X)} \; (i = \sqrt{-1}, \;\; X \in \mathfrak{h}).$$

- Let $\tau(f, H) = \mathrm{Ind}_H^G \chi$ be the induced (monomial) representation of G.

- Let $K(G,H,f)$ be the space of complex valued continuous fonctions φ on G, with compact support modulo H and which verifies

$$\varphi(gh) = \overline{\chi_f(h)}\varphi(g)$$

for all $g \in G$ and $h \in H$.

- The group G acts on $K(G,H,f)$ by left translation. For $\xi, \eta \in K(G,H,f)$, we have a G-invariant scalar product

$$\langle \xi, \eta \rangle = \oint_{G/H} \xi(g)\overline{\eta(g)}d\nu(g).$$

- The representation $\tau(f,H)$ is realized by left translation on the space $L^2(G/H,\chi_f)$, the completion of $K(G,H,f)$ with respect to the scalar product $\langle , \rangle$.

- Let $K(G, H, f)$ be the space of complex valued continuous fonctions φ on G, with compact support modulo H and which verifies

$$\varphi(gh) = \overline{\chi_f(h)}\varphi(g)$$

for all $g \in G$ and $h \in H$.

- The group G acts on $K(G, H, f)$ by left translation. For $\xi, \eta \in K(G, H, f)$, we have a G-invariant scalar product

$$\langle \xi, \eta \rangle = \oint_{G/H} \xi(g)\overline{\eta(g)}d\nu(g).$$

- The representation $\tau(f, H)$ is realized by left translation on the space $L^2(G/H, \chi_f)$, the completion of $K(G, H, f)$ with respect to the scalar product $\langle , \rangle$.

Kirillov Theory for nilpotent Lie groups

- Let $K(G,H,f)$ be the space of complex valued continuous fonctions φ on G, with compact support modulo H and which verifies

$$\varphi(gh) = \overline{\chi_f(h)}\varphi(g)$$

for all $g \in G$ and $h \in H$.

- The group G acts on $K(G,H,f)$ by left translation. For $\xi, \eta \in K(G,H,f)$, we have a G-invariant scalar product

$$\langle \xi, \eta \rangle = \oint_{G/H} \xi(g)\overline{\eta(g)} d\nu(g).$$

- The representation $\tau(f,H)$ is realized by left translation on the space $L^2(G/H,\chi_f)$, the completion of $K(G,H,f)$ with respect to the scalar product $\langle , \rangle$.

Université de Sfax
TUNISIE

- $\tau(f, H)$ is irreducible if and only if $\mathfrak{h} = \mathfrak{b}[f]$ is a polarization of f in $\mathfrak{g}$ (Lagrangian subalgebra of $\mathfrak{g}$ of maximal dimension).
- Any unitary and irreducible representation of G is obtained through this process.
- Let $B[f] = \exp \mathfrak{b}[f]$. The mapping $K : \mathfrak{g}^* \to \widehat{G}$, $f \mapsto \tau(f, B[f])$ is called the Kirillov-Bernat mapping which factors to a bijection $\tilde{K}$ through the quotient $\mathfrak{g}^*/Ad^*$:=Coadjoint orbit space.
- $\tilde{K} : \mathfrak{g}^*/Ad^* \to \widehat{G}$ is a homeomorphism.

Kirillov Theory for nilpotent Lie groups

- $\tau(f, H)$ is irreducible if and only if $\mathfrak{h} = \mathfrak{b}[f]$ is a polarization of f in $\mathfrak{g}$ (Lagrangian subalgebra of $\mathfrak{g}$ of maximal dimension).
- Any unitary and irreducible representation of G is obtained through this process.
- Let $B[f] = \exp \mathfrak{b}[f]$. The mapping $K : \mathfrak{g}^* \to \widehat{G}$, $f \mapsto \tau(f, B[f])$ is called the Kirillov-Bernat mapping which factors to a bijection $\tilde{K}$ through the quotient $\mathfrak{g}^*/Ad^*$:=Coadjoint orbit space.
- $\tilde{K} : \mathfrak{g}^*/Ad^* \to \widehat{G}$ is a homeomorphism.

Université de Sfax
TUNISIE

Kirillov Theory for nilpotent Lie groups

- $\tau(f, H)$ is irreducible if and only if $\mathfrak{h} = \mathfrak{b}[f]$ is a polarization of f in $\mathfrak{g}$ (Lagrangian subalgebra of $\mathfrak{g}$ of maximal dimension).
- Any unitary and irreducible representation of G is obtained through this process.
- Let $B[f] = \exp \mathfrak{b}[f]$. The mapping $K : \mathfrak{g}^* \to \widehat{G}$, $f \mapsto \tau(f, B[f])$ is called the Kirillov-Bernat mapping which factors to a bijection $\tilde{K}$ through the quotient $\mathfrak{g}^*/Ad^*$:=Coadjoint orbit space.
- $\tilde{K} : \mathfrak{g}^*/Ad^* \to \widehat{G}$ is a homeomorphism.

Université de Sfax
TUNISIE

Kirillov Theory for nilpotent Lie groups

- $\tau(f, H)$ is irreducible if and only if $\mathfrak{h} = \mathfrak{b}[f]$ is a polarization of f in $\mathfrak{g}$ (Lagrangian subalgebra of $\mathfrak{g}$ of maximal dimension).
- Any unitary and irreducible representation of G is obtained through this process.
- Let $B[f] = \exp \mathfrak{b}[f]$. The mapping $K : \mathfrak{g}^* \to \widehat{G}$, $f \mapsto \tau(f, B[f])$ is called the Kirillov-Bernat mapping which factors to a bijection $\tilde{K}$ through the quotient $\mathfrak{g}^*/Ad^*$:=Coadjoint orbit space.
- $\tilde{K} : \mathfrak{g}^*/Ad^* \to \widehat{G}$ is a homeomorphism.

Université de Sfax
TUNISIE

Disintegration of monomial representations: Branching laws

- Let $\tilde{\mu}$ be a positive measure on the affine space $\Gamma_f = f + \mathfrak{h}^\perp$ which is equivalent to the Lebesgue measure on Γ_f. Let μ be the image of $\tilde{\mu}$ by the Kirillov-Bernat $K : \mathfrak{g}^* \to \widehat{G}$.
- Then (Baklouti-Ludwig, Corwin-Greenleaf-Grelaud, Fujiwara, Lipsman):

$$\tau(f, H) \simeq \int_{\widehat{G}}^{\oplus} m(\pi)\pi d\mu(\pi),$$

where the multiplicity function is given as the number of H-orbits in $\Gamma_f \cap \Omega(\pi)$. Here, $\Omega(\pi)$ designates the coadjoint orbit associated to π.

- *Furthermore, $m(\pi)$ is the number of connected components of $\Gamma_f \cap \Omega(\pi)$ whenever each component is a manifold of dimension $\frac{\dim \Omega(\pi)}{2}$. Otherwise, $m(\pi)$ is equal to $+\infty$.*

Université de Sfax
TUNISIE

Disintegration of monomial representations: Branching laws

- Let $\tilde{\mu}$ be a positive measure on the affine space $\Gamma_f = f + \mathfrak{h}^\perp$ which is equivalent to the Lebesgue measure on Γ_f. Let μ be the image of $\tilde{\mu}$ by the Kirillov-Bernat $K : \mathfrak{g}^* \to \widehat{G}$.
- Then (Baklouti-Ludwig, Corwin-Greenleaf-Grelaud, Fujiwara, Lipsman):

$$\tau(f, H) \simeq \int_{\widehat{G}}^{\oplus} m(\pi)\pi d\mu(\pi),$$

where the multiplicity function is given as the number of H-orbits in $\Gamma_f \cap \Omega(\pi)$. Here, $\Omega(\pi)$ designates the coadjoint orbit associated to π.

- *Furthermore, $m(\pi)$ is the number of connected components of $\Gamma_f \cap \Omega(\pi)$ whenever each component is a manifold of dimension $\frac{\dim \Omega(\pi)}{2}$. Otherwise, $m(\pi)$ is equal to $+\infty$.*

Disintegration of monomial representations: Branching laws

- Let $\tilde{\mu}$ be a positive measure on the affine space $\Gamma_f = f + \mathfrak{h}^\perp$ which is equivalent to the Lebesgue measure on Γ_f. Let μ be the image of $\tilde{\mu}$ by the Kirillov-Bernat $K : \mathfrak{g}^* \to \widehat{G}$.
- Then (Baklouti-Ludwig, Corwin-Greenleaf-Grelaud, Fujiwara, Lipsman):

$$\tau(f, H) \simeq \int_{\widehat{G}}^{\oplus} m(\pi)\pi d\mu(\pi),$$

where the multiplicity function is given as the number of H-orbits in $\Gamma_f \cap \Omega(\pi)$. Here, $\Omega(\pi)$ designates the coadjoint orbit associated to π.

- *Furthermore, $m(\pi)$ is the number of connected components of $\Gamma_f \cap \Omega(\pi)$ whenever each component is a manifold of dimension $\frac{\dim \Omega(\pi)}{2}$. Otherwise, $m(\pi)$ is equal to $+\infty$.*

Disintegration of monomial representations: Branching laws

❍ Problems related to multiplicities

(1) The multiplicity $m(\pi)$ (as a function of π) is always uniformly infinite or ($\mu-$almost) everywhere finite.

(2) A necessary and sufficient condition for finiteness is that for generic $\ell \in \Gamma_f$, we have $\dim G \cdot \ell = 2 \dim H \cdot \ell$.

(3) In the finite case, the multiplicity is bounded.

(4) In the finite case, the parity of the multiplicity is constant.

Université de Sfax
TUNISIE

Disintegration of monomial representations: Branching laws

❍ Problems related to multiplicities

(1) The multiplicity $m(\pi)$ (as a function of π) is always uniformly infinite or ($\mu-$almost) everywhere finite.

(2) A necessary and sufficient condition for finiteness is that for generic $\ell \in \Gamma_f$, we have $\dim G \cdot \ell = 2 \dim H \cdot \ell$.

(3) In the finite case, the multiplicity is bounded.

(4) In the finite case, the parity of the multiplicity is constant.

Disintegration of monomial representations: Branching laws

❍ Problems related to multiplicities

(1) The multiplicity $m(\pi)$ (as a function of π) is always uniformly infinite or ($\mu-$almost) everywhere finite.

(2) A necessary and sufficient condition for finiteness is that for generic $\ell \in \Gamma_f$, we have $\dim G \cdot \ell = 2 \dim H \cdot \ell$.

(3) In the finite case, the multiplicity is bounded.

(4) In the finite case, the parity of the multiplicity is constant.

Disintegration of monomial representations: Branching laws

❍ Problems related to multiplicities

(1) The multiplicity $m(\pi)$ (as a function of π) is always uniformly infinite or ($\mu-$almost) everywhere finite.

(2) A necessary and sufficient condition for finiteness is that for generic $\ell \in \Gamma_f$, we have $\dim G \cdot \ell = 2 \dim H \cdot \ell$.

(3) In the finite case, the multiplicity is bounded.

(4) In the finite case, the parity of the multiplicity is constant.

Université de Sfax
TUNISIE

Branching laws of restrictions of representations

- Let $K = \exp \mathfrak{k}$ be a closed connected Lie subgroup of G.
- Let $p : \mathfrak{g}^* \to \mathfrak{k}^*$ be the restriction mapping.
- For $\pi \in \hat{G}$, we consider a finite measure μ_π on $\mathfrak{g}^*$ equivalent to the canonical measure on the orbit $\Omega_G(\pi)$ which is regarded as a measure on $\mathfrak{g}^*$.
- Put $\nu_\pi = (\theta_K \circ p)_*(\mu_\pi)$. The restriction $\pi|_K$ of π to K is disintegrated as:

$$\pi|_K \simeq \int_{\hat{K}}^{\oplus} m_\sigma \sigma \, d\nu_\pi(\sigma)$$

- Let $K = \exp \mathfrak{k}$ be a closed connected Lie subgroup of G.
- Let $p : \mathfrak{g}^* \to \mathfrak{k}^*$ be the restriction mapping.
- For $\pi \in \hat{G}$, we consider a finite measure μ_π on $\mathfrak{g}^*$ equivalent to the canonical measure on the orbit $\Omega_G(\pi)$ which is regarded as a measure on $\mathfrak{g}^*$.
- Put $\nu_\pi = (\theta_K \circ p)_*(\mu_\pi)$. The restriction $\pi|_K$ of π to K is disintegrated as:

$$\pi|_K \simeq \int_{\hat{K}}^{\oplus} m_\sigma^\pi \sigma \, d\nu_\pi(\sigma)$$

Université de Sfax
TUNISIE

- Let $K = \exp \mathfrak{k}$ be a closed connected Lie subgroup of G.
- Let $p : \mathfrak{g}^* \to \mathfrak{k}^*$ be the restriction mapping.
- For $\pi \in \hat{G}$, we consider a finite measure μ_π on $\mathfrak{g}^*$ equivalent to the canonical measure on the orbit $\Omega_G(\pi)$ which is regarded as a measure on $\mathfrak{g}^*$.
- Put $\nu_\pi = (\theta_K \circ p)_*(\mu_\pi)$. The restriction $\pi|_K$ of π to K is disintegrated as:

$$\pi|_K \simeq \int_{\hat{K}}^{\oplus} m_\sigma^\pi \sigma d\nu_\pi(\sigma)$$

Université de Sfax TUNISIE

Branching laws of restrictions of representations

- Let $K = \exp \mathfrak{k}$ be a closed connected Lie subgroup of G.

- Let $p : \mathfrak{g}^* \to \mathfrak{k}^*$ be the restriction mapping.

- For $\pi \in \hat{G}$, we consider a finite measure μ_π on $\mathfrak{g}^*$ equivalent to the canonical measure on the orbit $\Omega_G(\pi)$ which is regarded as a measure on $\mathfrak{g}^*$.

- Put $\nu_\pi = (\theta_K \circ p)_*(\mu_\pi)$. The restriction $\pi|_K$ of π to K is disintegrated as:

$$\pi|_K \simeq \int_{\hat{K}}^{\oplus} m_\sigma^\pi \sigma d\nu_\pi(\sigma)$$

- The multiplicities m_σ^π is obtained as the number of the K-orbits contained in $\Omega_G(\pi) \cap p^{-1}(\Omega_K(\sigma))$.

- Then μ_π-almost everywhere, the multiplicities are either uniformly bounded or equal to infinity

- According to these two eventualities, we say that the representation $\pi|_K$ has either finite or infinite multiplicities.

- The multiplicities m_σ^π is obtained as the number of the K-orbits contained in $\Omega_G(\pi) \cap p^{-1}(\Omega_K(\sigma))$.

- Then μ_π-almost everywhere, the multiplicities are either uniformly bounded or equal to infinity

- According to these two eventualities, we say that the representation $\pi|_K$ has either finite or infinite multiplicities.

Université de Sfax
TUNISIE

Branching laws of restrictions of representations

- The multiplicities m_σ^π is obtained as the number of the K-orbits contained in $\Omega_G(\pi) \cap p^{-1}(\Omega_K(\sigma))$.

- Then μ_π-almost everywhere, the multiplicities are either uniformly bounded or equal to infinity

- According to these two eventualities, we say that the representation $\pi|_K$ has either finite or infinite multiplicities.

Université de Sfax
TUNISIE

An algebra of differential operators

- Let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ and let $\ker(\pi)$ be the primitive ideal of $\mathcal{U}(\mathfrak{g})$ associated to π.

- We introduce the algebra

$$\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} = \{A \in \mathcal{U}(\mathfrak{g}); [A, \mathfrak{k}] \subset \ker(\pi)\}$$

and its image

$$D_\pi(G)^K \cong \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}/\ker(\pi) \cong (\mathcal{U}(\mathfrak{g})/\ker(\pi))^K,$$

where the last algebra designates the quotient algebra of K-invariant elements.

- $D_\pi(G)^K$ is nothing but an algebra of differential opertors with polynomial coefficients on the space of C^∞-vectors of π.

An algebra of differential operators

- Let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ and let $\ker(\pi)$ be the primitive ideal of $\mathcal{U}(\mathfrak{g})$ associated to π.

- We introduce the algebra

$$\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} = \{A \in \mathcal{U}(\mathfrak{g}); [A, \mathfrak{k}] \subset \ker(\pi)\}$$

and its image

$$D_\pi(G)^K \cong \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}/\ker(\pi) \cong (\mathcal{U}(\mathfrak{g})/\ker(\pi))^K,$$

where the last algebra designates the quotient algebra of K-invariant elements.

- $D_\pi(G)^K$ is nothing but an algebra of differential opertors with polynomial coefficients on the space of C^∞-vectors of π.

Université de Sfax
TUNISIE

An algebra of differential operators

❍ Let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ and let $\ker(\pi)$ be the primitive ideal of $\mathcal{U}(\mathfrak{g})$ associated to π.

❍ We introduce the algebra

$$\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} = \{A \in \mathcal{U}(\mathfrak{g}); [A, \mathfrak{k}] \subset \ker(\pi)\}$$

and its image

$$D_\pi(G)^K \cong \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}/\ker(\pi) \cong (\mathcal{U}(\mathfrak{g})/\ker(\pi))^K,$$

where the last algebra designates the quotient algebra of K-invariant elements.

❍ $D_\pi(G)^K$ is nothing but an algebra of differential opertors with polynomial coefficients on the space of C^∞-vectors of π.

Two Conjectures: Corwin-Greenleaf(1992), Baklouti-Fujiwara(2004)

Conjecture 1 (Commutativity Conjecture): *Let G be a connected and simply connected nilpotent Lie group, K an analytic subgroup of G. Then the algebra $D_\pi(G)^K$ is commutative if and only if the representation $\pi|_K$ has finite multiplicities.*

Université de Sfax TUNISIE

Two Conjectures: Corwin-Greenleaf(1992), Baklouti-Fujiwara(2004)

Conjecture 2 (Polynomial Conjecture): *Let G be a connected and simply connected nilpotent Lie group, K an analytic subgroup of G. Let $\pi \in \hat{G}$ be a unitary and irreducible representation of G such that $\pi|_K$ is of finite multiplicities. Then the algebra $D_\pi(G)^K$ is isomorphic to the algebra $\mathbb{C}[\Omega(\pi)]^K$ of the K-invariant polynomial functions on $\Omega(\pi)$.*

Université de Sfax
TUNISIE

- Introducing some algebraic tools to describe the generators of the algebra $D_\pi(G)^K$ in term of the envelopping algebra of $\mathfrak{g}_\mathbb{C}$, we proved Conjecture 1:

Theorem 1 (A. Bak, H. Fujiwara, 2004)

Conjecture 1 holds in the setting of connected simply connected nilpotent Lie groups.

- This makes use of Pedersen's construction of the kernel $\ker(\pi)$, π being the Kirillov's model associated to $\Omega(\pi)$

Université de Sfax
TUNISIE

About Conjecture 1

❍ Introducing some algebraic tools to describe the generators of the algebra $D_\pi(G)^K$ in term of the enveloping algebra of $\mathfrak{g}_\mathbb{C}$, we proved Conjecture 1:

Theorem 1 (A. Bak, H. Fujiwara, 2004)

Conjecture 1 holds in the setting of connected simply connected nilpotent Lie groups.

❍ This makes use of Pedersen's construction of the kernel $\ker(\pi)$, π being the Kirillov's model associated to $\Omega(\pi)$

Université de Sfax
TUNISIE

❍ Introducing some algebraic tools to describe the generators of the algebra $D_\pi(G)^K$ in term of the envelopping algebra of $\mathfrak{g}_\mathbb{C}$, we proved Conjecture 1:

Theorem 1 (A. Bak, H. Fujiwara, 2004)

Conjecture 1 holds in the setting of connected simply connected nilpotent Lie groups.

❍ This makes use of Pedersen's construction of the kernel $\ker(\pi)$, π being the Kirillov's model associated to $\Omega(\pi)$

Université de Sfax
TUNISIE

About Conjecture 2: Longstanding joint works with H. Fujiwara and J. Ludwig

- We positively proved Conjecture 2 in many settings:
- The case where K is a normal subgroup of G or where the orbit $\Omega(\pi)$ is flat (Bull. Sci. Math, 2005).
- K is abelian or where $\Omega(\pi)$ admits a normal polarizing subgroup (J. Lie. Theory, 2019).

Université de Sfax
TUNISIE

About Conjecture 2: Longstanding joint works with H. Fujiwara and J. Ludwig

- We positively proved Conjecture 2 in many settings:
- The case where K is a normal subgroup of G or where the orbit $\Omega(\pi)$ is flat (Bull. Sci. Math, 2005).
- K is abelian or where $\Omega(\pi)$ admits a normal polarizing subgroup (J. Lie. Theory, 2019).

Université de Sfax
TUNISIE

About Conjecture 2: Longstanding joint works with H. Fujiwara and J. Ludwig

- We positively proved Conjecture 2 in many settings:
- The case where K is a normal subgroup of G or where the orbit $\Omega(\pi)$ is flat (Bull. Sci. Math, 2005).
- K is abelian or where $\Omega(\pi)$ admits a normal polarizing subgroup (J. Lie. Theory, 2019).

Université de Sfax
TUNISIE

About Conjecture 2: Rationality

- Let $\ell \in \Omega(\pi)$ and $\mathfrak{b}[\ell]$ a real polarization at ℓ of $\mathfrak{g}$. Realize π as $\pi = \mathrm{ind}_{B[\ell]}^{G} \chi_\ell$ with $B[\ell] = \exp(\mathfrak{b}[\ell])$ and χ_ℓ is the unitary character of $B[\ell]$ whose differential is $i\ell|_{\mathfrak{b}[\ell]}$.

- We also construct a real polarization $\mathfrak{b}[\ell|_{\mathfrak{k}}]$ of $\mathfrak{k}$ at $\ell|_{\mathfrak{k}} \in \mathfrak{k}^*$. Put $B[\ell|_{\mathfrak{k}}] = \exp(\mathfrak{b}[\ell|_{\mathfrak{k}}])$.

- Let for $\varphi \in \mathcal{H}_\pi^\infty$, the semi-invariant generalized Penney distribution:

$$\langle a_\ell^K, \varphi \rangle = \int_{B[\ell|_{\mathfrak{k}}]/(B[\ell|_{\mathfrak{k}}] \cap B[\ell])} \overline{\varphi(b)\chi_\ell(b)}\, d\dot{b},$$

- $d\dot{b}$ designating an invariant measure on the homogeneous space $B[\ell|_{\mathfrak{k}}]/(B[\ell|_{\mathfrak{k}}] \cap B[\ell])$

Université de Sfax
TUNISIE

About Conjecture 2: Rationality

- Let $\ell \in \Omega(\pi)$ and $\mathfrak{b}[\ell]$ a real polarization at ℓ of $\mathfrak{g}$. Realize π as $\pi = \mathrm{ind}_{B[\ell]}^{G}\chi_\ell$ with $B[\ell] = \exp(\mathfrak{b}[\ell])$ and χ_ℓ is the unitary character of $B[\ell]$ whose differential is $i\ell|_{\mathfrak{b}[\ell]}$.

- We also construct a real polarization $\mathfrak{b}[\ell|_{\mathfrak{k}}]$ of $\mathfrak{k}$ at $\ell|_{\mathfrak{k}} \in \mathfrak{k}^*$. Put $B[\ell|_{\mathfrak{k}}] = \exp(\mathfrak{b}[\ell|_{\mathfrak{k}}])$.

- Let for $\varphi \in \mathcal{H}_\pi^\infty$, the semi-invariant generalized Penney distribution:

$$\langle a_\ell^K, \varphi \rangle = \int_{B[\ell|_{\mathfrak{k}}]/(B[\ell|_{\mathfrak{k}}]\cap B[\ell])} \overline{\varphi(b)\chi_\ell(b)}\, d\dot{b},$$

- $d\dot{b}$ designating an invariant measure on the homogeneous space $B[\ell|_{\mathfrak{k}}]/(B[\ell|_{\mathfrak{k}}] \cap B[\ell])$

Université de Sfax TUNISIE

About Conjecture 2: Rationality

❍ Let $\ell \in \Omega(\pi)$ and $\mathfrak{b}[\ell]$ a real polarization at ℓ of $\mathfrak{g}$. Realize π as $\pi = \mathrm{ind}_{B[\ell]}^{G} \chi_\ell$ with $B[\ell] = \exp(\mathfrak{b}[\ell])$ and χ_ℓ is the unitary character of $B[\ell]$ whose differential is $i\ell|_{\mathfrak{b}[\ell]}$.

❍ We also construct a real polarization $\mathfrak{b}[\ell|_{\mathfrak{k}}]$ of $\mathfrak{k}$ at $\ell|_{\mathfrak{k}} \in \mathfrak{k}^*$. Put $B[\ell|_{\mathfrak{k}}] = \exp(\mathfrak{b}[\ell|_{\mathfrak{k}}])$.

❍ Let for $\varphi \in \mathcal{H}_\pi^\infty$, the semi-invariant generalized Penney distribution:

$$\langle a_\ell^K, \varphi \rangle = \int_{B[\ell|_{\mathfrak{k}}]/(B[\ell|_{\mathfrak{k}}] \cap B[\ell])} \overline{\varphi(b)\chi_\ell(b)} d\dot{b},$$

❍ $d\dot{b}$ designating an invariant measure on the homogeneous space $B[\ell|_{\mathfrak{k}}]/(B[\ell|_{\mathfrak{k}}] \cap B[\ell])$

Université de Sfax
TUNISIE

About Conjecture 2: Rationality

- Let $\ell \in \Omega(\pi)$ and $\mathfrak{b}[\ell]$ a real polarization at ℓ of $\mathfrak{g}$. Realize π as $\pi = \mathrm{ind}_{B[\ell]}^{G} \chi_\ell$ with $B[\ell] = \exp(\mathfrak{b}[\ell])$ and χ_ℓ is the unitary character of $B[\ell]$ whose differential is $i\ell|_{\mathfrak{b}[\ell]}$.

- We also construct a real polarization $\mathfrak{b}[\ell|_{\mathfrak{k}}]$ of $\mathfrak{k}$ at $\ell|_{\mathfrak{k}} \in \mathfrak{k}^*$. Put $B[\ell|_{\mathfrak{k}}] = \exp(\mathfrak{b}[\ell|_{\mathfrak{k}}])$.

- Let for $\varphi \in \mathcal{H}_\pi^\infty$, the semi-invariant generalized Penney distribution:

$$\langle a_\ell^K, \varphi \rangle = \int_{B[\ell|_{\mathfrak{k}}]/(B[\ell|_{\mathfrak{k}}] \cap B[\ell])} \overline{\varphi(b)\chi_\ell(b)}\, d\dot{b},$$

- $d\dot{b}$ designating an invariant measure on the homogeneous space $B[\ell|_{\mathfrak{k}}]/(B[\ell|_{\mathfrak{k}}] \cap B[\ell])$

About Conjecture 2: Rationality

- Then μ_π-almost everywhere in $\Omega(\pi)$, the vector $a_\ell^K \in \mathcal{H}_\pi^{-\infty}$ is an eigen-vector for all the elements of $D_\pi(G)^K$ acting on $\mathcal{H}_\pi^\infty$ by continuity.

- This means that for any $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$, we have

$$W \cdot a_\ell^K := \pi(W) a_\ell^K = P_W(\ell) a_\ell^K$$

with a certain complex scalar $P_W(\ell)$.

- The scalar $P_W(\ell)$ does not depend on the choice of polarizations.

- Then μ_π-almost everywhere in $\Omega(\pi)$, the vector $a_\ell^K \in \mathcal{H}_\pi^{-\infty}$ is an eigen-vector for all the elements of $D_\pi(G)^K$ acting on $\mathcal{H}_\pi^\infty$ by continuity.

- This means that for any $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$, we have

$$W \cdot a_\ell^K := \pi(W)a_\ell^K = P_W(\ell)a_\ell^K$$

with a certain complex scalar $P_W(\ell)$.

- The scalar $P_W(\ell)$ does not depend on the choice of polarizations.

Université de Sfax TUNISIE

- Then μ_π-almost everywhere in $\Omega(\pi)$, the vector $a_\ell^K \in \mathcal{H}_\pi^{-\infty}$ is an eigen-vector for all the elements of $D_\pi(G)^K$ acting on $\mathcal{H}_\pi^\infty$ by continuity.

- This means that for any $W \in \mathcal{U}_\pi(\mathfrak{g})^\mathfrak{k}$, we have

$$W \cdot a_\ell^K := \pi(W) a_\ell^K = P_W(\ell) a_\ell^K$$

with a certain complex scalar $P_W(\ell)$.

- The scalar $P_W(\ell)$ does not depend on the choice of polarizations.

Université de Sfax
TUNISIE

About Conjecture 2: Rationality

Theorem 2

Suppose that $\pi_{|K}$ has finite multiplicities. The homomorphism $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \ni W \mapsto P_W : \ell \mapsto P_W(\ell)$ defines an imbedding of $D_\pi(G)^K$ into the field $\mathbb{C}(\Omega(\pi))^K$ of rational K-invariant functions on $\Omega(\pi)$.

- Is it true that $\ell \mapsto P_W(\ell)$ continuously extends to a K-invariant polynomial function on $\Omega(\pi)$?
- The fact that $\pi_{|K}$ has finite multiplicities complicates the complex induction procedure.

Université de Sfax
TUNISIE

About Conjecture 2: Rationality

Theorem 2

Suppose that $\pi_{|K}$ has finite multiplicities. The homomorphism $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \ni W \mapsto P_W : \ell \mapsto P_W(\ell)$ defines an imbedding of $D_\pi(G)^K$ into the field $\mathbb{C}(\Omega(\pi))^K$ of rational K-invariant functions on $\Omega(\pi)$.

- Is it true that $\ell \mapsto P_W(\ell)$ continuously extends to a K-invariant polynomial function on $\Omega(\pi)$?
- The fact that $\pi_{|K}$ has finite multiplicities complicates the complex induction procedure.

About Conjecture 2: Rationality

Theorem 2

Suppose that $\pi_{|K}$ has finite multiplicities. The homomorphism $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} \ni W \mapsto P_W : \ell \mapsto P_W(\ell)$ defines an imbedding of $D_\pi(G)^K$ into the field $\mathbb{C}(\Omega(\pi))^K$ of rational K-invariant functions on $\Omega(\pi)$.

- Is it true that $\ell \mapsto P_W(\ell)$ continuously extends to a K-invariant polynomial function on $\Omega(\pi)$?

- The fact that $\pi_{|K}$ has finite multiplicities complicates the complex induction procedure.

About Conjecture 2: K-diagonal elements

- An element $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is said to be K-diagonal, if

$$W \cdot a_\ell^K = P_W(\ell) a_\ell^K$$

for a certain scalar $P_W(\ell) \in \mathbb{C}$ independent of the choice of polarizations to describe the distribution a_ℓ^K and $\ell \mapsto P_W(\ell)$ extends to a rational function on $\Omega(\pi)$.

- Any K-diagonal element of $\mathcal{U}(\mathfrak{g})$ belongs to $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$.

- In the case where $\pi|_K$ has finite multiplicities, any element of $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is K-diagonal.

- Let $W \in \mathcal{U}(\mathfrak{g})$ be K-diagonal. Then P_W is identically zero if and only if $W \in \ker(\pi)$.

About Conjecture 2: K-diagonal elements

- An element $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is said to be K-diagonal, if

$$W \cdot a_\ell^K \quad = \quad P_W(\ell) a_\ell^K$$

for a certain scalar $P_W(\ell) \in \mathbb{C}$ independent of the choice of polarizations to describe the distribution a_ℓ^K and $\ell \mapsto P_W(\ell)$ extends to a rational function on $\Omega(\pi)$.

- Any K-diagonal element of $\mathcal{U}(\mathfrak{g})$ belongs to $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$.

- In the case where $\pi|_K$ has finite multiplicities, any element of $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is K-diagonal.

- Let $W \in \mathcal{U}(\mathfrak{g})$ be K-diagonal. Then P_W is identically zero if and only if $W \in \ker(\pi)$.

Université de Sfax
TUNISIE

About Conjecture 2: K-diagonal elements

- An element $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is said to be K-diagonal, if

$$W \cdot a_\ell^K = P_W(\ell) a_\ell^K$$

for a certain scalar $P_W(\ell) \in \mathbb{C}$ independent of the choice of polarizations to describe the distribution a_ℓ^K and $\ell \mapsto P_W(\ell)$ extends to a rational function on $\Omega(\pi)$.

- Any K-diagonal element of $\mathcal{U}(\mathfrak{g})$ belongs to $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$.

- In the case where $\pi|_K$ has finite multiplicities, any element of $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is K-diagonal.

- Let $W \in \mathcal{U}(\mathfrak{g})$ be K-diagonal. Then P_W is identically zero if and only if $W \in \ker(\pi)$.

Université de Sfax
TUNISIE

- An element $W \in \mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is said to be K-diagonal, if

$$W \cdot a_\ell^K = P_W(\ell) a_\ell^K$$

for a certain scalar $P_W(\ell) \in \mathbb{C}$ independent of the choice of polarizations to describe the distribution a_ℓ^K and $\ell \mapsto P_W(\ell)$ extends to a rational function on $\Omega(\pi)$.

- Any K-diagonal element of $\mathcal{U}(\mathfrak{g})$ belongs to $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$.

- In the case where $\pi|_K$ has finite multiplicities, any element of $\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is K-diagonal.

- Let $W \in \mathcal{U}(\mathfrak{g})$ be K-diagonal. Then P_W is identically zero if and only if $W \in \ker(\pi)$.

Université de Sfax
TUNISIE

About Conjecture 2: K-diagonal elements

Theorem 3

Let $\pi \in \hat{G}$ and let $W \in \mathcal{U}(\mathfrak{g})$ be K-diagonal. The function P_W extends to a K-invariant polynomial function on $\Omega(\pi)$.

- The geometries of K and G-coadjoint orbits play a crucial role in the proof (in term of saturation with respect to one codimensional ideals).

Université de Sfax
TUNISIE

About Conjecture 2: K-diagonal elements

Theorem 3

Let $\pi \in \hat{G}$ and let $W \in \mathcal{U}(\mathfrak{g})$ be K-diagonal. The function P_W extends to a K-invariant polynomial function on $\Omega(\pi)$.

- The geometries of K and G-coadjoint orbits play a crucial role in the proof (in term of saturation with respect to one codimensional ideals).

- Let $\mathcal{V}$ be the set of K-diagonal elements. Consider the mapping

$$\Theta : \mathcal{V} \ni W \mapsto P_W \in \mathbb{C}[\Omega(\pi)]^K.$$

Theorem 4

Assume that $\pi|_K$ has finite multiplicities. Then the mapping Θ is surjective.

Université de Sfax
TUNISIE

Conjecture 2 holds

- And finally:

Theorem 5 (RT-AMS, 2022)

Conjecture 2 holds in the setting of connected simply connected nilpotent Lie groups.

- For $A \in \mathcal{U}(\mathfrak{g})$, we denote by $R(A)$ its right action. Consider back $\tau := \tau(f, H)$.

- Define the vector subspace $\mathfrak{a}_\tau$ of $\mathcal{U}(\mathfrak{g})$ to be the span of the set $\{Y + if(Y), \ Y \in \mathfrak{h}\}$.

- Let $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ be the left ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{a}_\tau$.

- $\mathcal{U}(\mathfrak{g}, \tau)$ to be the collection of all $A \in \mathcal{U}(\mathfrak{g})$ such that $R(A)$ leaves $C^\infty(G, \tau)$ stable, where:

$$C^\infty(G,\tau) = \{\varphi : G \to \mathbb{C}; \varphi \in C^\infty(G), \varphi(gh) = \overline{\chi(h)}\varphi(g), \; g \in G, h \in H\}.$$

- For $A \in \mathcal{U}(\mathfrak{g})$, we denote by $R(A)$ its right action. Consider back $\tau := \tau(f, H)$.

- Define the vector subspace $\mathfrak{a}_\tau$ of $\mathcal{U}(\mathfrak{g})$ to be the span of the set $\{Y + if(Y),\ Y \in \mathfrak{h}\}$.

- Let $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ be the left ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{a}_\tau$.

- $\mathcal{U}(\mathfrak{g}, \tau)$ to be the collection of all $A \in \mathcal{U}(\mathfrak{g})$ such that $R(A)$ leaves $C^\infty(G, \tau)$ stable, where:

$$C^\infty(G,\tau) = \{\varphi : G \to \mathbb{C}; \varphi \in C^\infty(G), \varphi_\ell(gh) = \overline{\chi(h)}\varphi(g), g \in G, h \in H\}.$$

The Corwin-Greenleaf Conjecture

- For $A \in \mathcal{U}(\mathfrak{g})$, we denote by $R(A)$ its right action. Consider back $\tau := \tau(f, H)$.

- Define the vector subspace $\mathfrak{a}_\tau$ of $\mathcal{U}(\mathfrak{g})$ to be the span of the set $\{Y + if(Y),\ Y \in \mathfrak{h}\}$.

- Let $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ be the left ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{a}_\tau$.

- $\mathcal{U}(\mathfrak{g}, \tau)$ to be the collection of all $A \in \mathcal{U}(\mathfrak{g})$ such that $R(A)$ leaves $C^\infty(G, \tau)$ stable, where:

$$C^\infty(G, \tau) = \{\varphi : G \to \mathbb{C}; \varphi \in C^\infty(G), \varphi_\ell(gh) = \overline{\chi(h)}\varphi(g),$$
$$g \in G, h \in H\}.$$

Université de Sfax
TUNISIE

The Corwin-Greenleaf Conjecture

- For $A \in \mathcal{U}(\mathfrak{g})$, we denote by $R(A)$ its right action. Consider back $\tau := \tau(f, H)$.

- Define the vector subspace $\mathfrak{a}_\tau$ of $\mathcal{U}(\mathfrak{g})$ to be the span of the set $\{Y + if(Y),\ \ Y \in \mathfrak{h}\}$.

- Let $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$ be the left ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{a}_\tau$.

- $\mathcal{U}(\mathfrak{g}, \tau)$ to be the collection of all $A \in \mathcal{U}(\mathfrak{g})$ such that $R(A)$ leaves $C^\infty(G, \tau)$ stable, where:

$$C^\infty(G, \tau) = \{\varphi : G \to \mathbb{C}; \varphi \in C^\infty(G), \varphi_\ell(gh) = \overline{\chi(h)}\varphi(g),$$

$$g \in G, h \in H\}.$$

- We then find that

$$\mathcal{U}(\mathfrak{g},\tau) = \{A \in \mathcal{U}(\mathfrak{g}); [Y, A] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau, \forall\ Y \in \mathfrak{h}\}.$$

- The map $A \mapsto R(A)|C^\infty(G,\tau)$ gives us an algebra homomorphism Φ of $\mathcal{U}(\mathfrak{g},\tau)$ onto $D_\tau(G/H)$ with kernel $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

- This means that

$$D_\tau(G/H) \cong \mathcal{U}(\mathfrak{g},\tau)/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau.$$

- We then find that

$$\mathcal{U}(\mathfrak{g},\tau) = \{A \in \mathcal{U}(\mathfrak{g}); [Y, A] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau, \forall\ Y \in \mathfrak{h}\}.$$

- The map $A \mapsto R(A)|C^\infty(G,\tau)$ gives us an algebra homomorphism Φ of $\mathcal{U}(\mathfrak{g},\tau)$ onto $D_\tau(G/H)$ with kernel $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

- This means that

$$D_\tau(G/H) \cong \mathcal{U}(\mathfrak{g},\tau)/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau.$$

Université de Sfax TUNISIE

- We then find that

$$\mathcal{U}(\mathfrak{g},\tau) = \{A \in \mathcal{U}(\mathfrak{g}); [Y, A] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau, \forall\ Y \in \mathfrak{h}\}.$$

- The map $A \mapsto R(A)|C^\infty(G,\tau)$ gives us an algebra homomorphism Φ of $\mathcal{U}(\mathfrak{g},\tau)$ onto $D_\tau(G/H)$ with kernel $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau$.

- This means that

$$D_\tau(G/H) \cong \mathcal{U}(\mathfrak{g},\tau)/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\tau.$$

Université de Sfax
TUNISIE

- We prove the Corwin-Greenleaf Conjecture:

Theorem 6 (Trans-AMS, 2023)

Let G be a connected and simply connected nilpotent Lie group, H an analytic subgroup of G and χ a unitary character of H. When τ has finite multiplicities, the algebra $D_\tau(G/H)$ is isomorphic to the algebra $\mathbb{C}[\Gamma_f]^H$ of the H-invariant polynomial functions on the affine space $\Gamma_f = f + \mathfrak{h}^\perp$.

Université de Sfax
TUNISIE

- And as a direct consequence:

Theorem 7

Suppose that τ has finite multiplicities. Then, any non-zero element of $D_\tau(G/H)$ admits a fundamental tempered solution.

A Counterexample: Negative answer to Duflo's Question

- Let

$$G = \left\{ \begin{pmatrix} a^2 & x & z \\ 0 & a & y \\ 0 & 0 & 1 \end{pmatrix} ; a > 0,\ x, y, z \in \mathbb{R} \right\}.$$

Then G is exponential solvable (but not nilpotent).

- Its Lie algebra $\mathfrak{g} = \langle T, X, Y, Z \rangle_{\mathbb{R}}$:

$$[T, X] = X, [T, Y] = Y, [T, Z] = 2Z, [X, Y] = Z.$$

A Counterexample: Negative answer to Duflo's Question

- Let

$$G = \left\{ \begin{pmatrix} a^2 & x & z \\ 0 & a & y \\ 0 & 0 & 1 \end{pmatrix} ; a > 0,\ \ x, y, z \in \mathbb{R} \right\}.$$

Then G is exponential solvable (but not nilpotent).

- Its Lie algebra $\mathfrak{g} = \langle T, X, Y, Z \rangle_{\mathbb{R}}$:

$$[T, X] = X, [T, Y] = Y, [T, Z] = 2Z, [X, Y] = Z.$$

A Counterexample: Negative answer to Duflo's Question

- Let $f = 0 \in \mathfrak{g}^*$ and $\mathfrak{h} = \mathbb{R}T$.
- $\tau = \mathrm{ind}_H^G 1$ is of infinite multiplicities, but $D_\tau(G/H)$ is trivial.
- This provides a negative answer to Duflo's Question, stating that there is equivalence between the fact that $D_\tau(G/H)$ and that the multiplicity of τ is of discrete type.

Université de Sfax
TUNISIE

A Counterexample: Negative answer to Duflo's Question

- Let $f = 0 \in \mathfrak{g}^*$ and $\mathfrak{h} = \mathbb{R}T$.
- $\tau = \mathrm{ind}_H^G 1$ is of infinite multiplicities, but $D_\tau(G/H)$ is trivial.
- This provides a negative answer to Duflo's Question, stating that there is equivalence between the fact that $D_\tau(G/H)$ and that the multiplicity of τ is of discrete type.

Université de Sfax
TUNISIE

A Counterexample: Negative answer to Duflo's Question

- Let $f = 0 \in \mathfrak{g}^*$ and $\mathfrak{h} = \mathbb{R}T$.
- $\tau = \mathrm{ind}_H^G 1$ is of infinite multiplicities, but $D_\tau(G/H)$ is trivial.
- This provides a negative answer to Duflo's Question, stating that there is equivalence between the fact that $D_\tau(G/H)$ and that the multiplicity of τ is of discrete type.

Université de Sfax
TUNISIE

- Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$ and $\mathfrak{a}_\tau = \{X + \sqrt{-1} f(X); X \in \mathfrak{h}\}$. Then, $S(\mathfrak{g})$ possesses the Poisson structure $\{,\}$ well determined by the equality $\{X, Y\} = [X, Y]$ if X, Y are in $\mathfrak{g}$.

- We consider the algebra $(S(\mathfrak{g})/S(\mathfrak{g})\overline{\mathfrak{a}_\tau})^H$ of the H-invariant polynomial functions on the affine subspace $\Gamma_\tau = \{\ell \in \mathfrak{g}^* : \ell(X) = f(X), X \in \mathfrak{h}\}$ of $\mathfrak{g}^*$. This inherits the Poisson structure from $S(\mathfrak{g})$.

- We denote by Z_τ its Poisson center and C_τ the center of $D_\tau(G/H)$. We here provide a positive solution to a problem due to Duflo stating that the algebras Z_τ and C_τ of $D_\tau(G/H)$ are isomorphic. Only some particular cases are treated so far (cf. Tanimura).

Université de Sfax
TUNISIE

The restriction case

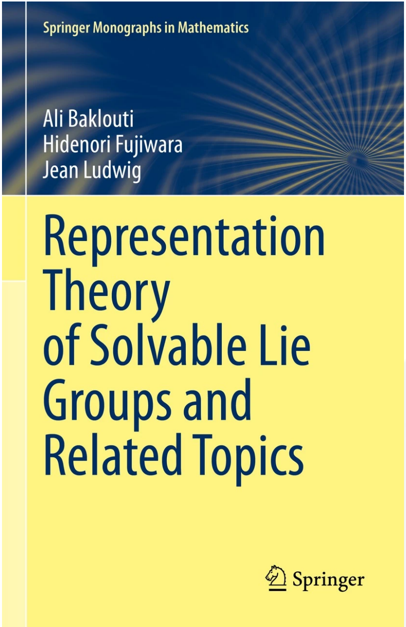
❍ Let $G = \exp \mathfrak{g}$ be a connected and simply connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$, $K = \exp \mathfrak{k}$ an analytic subgroup of G with Lie algebra $\mathfrak{k}$, π an irreducible unitary representation of G and $\pi|_K$ the restriction of π to K.

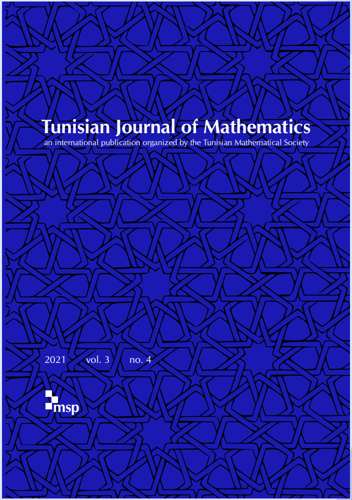
❍ Let $D_\pi(G)^K$ be the algebra of the differential operators keeping invariant the space of $C^{+\infty}$-vectors of π and commuting with the action of K on that space.

❍ Let Ω be the coadjoint orbit of G corresponding to π and $\mathbb{C}[\Omega]^K$ the algebra of K-invariant polynomial functions on Ω. Remark that $\mathbb{C}[\Omega]^K$ is endowed with a Poisson product coming from the symplectic structure of Ω.

❍ We show that the center of $D_\pi(G)^K$ and the Poisson center of $\mathbb{C}[\Omega]^K$ are isomorphic.

Université de Sfax
TUNISIE


Springer Monographs in Mathematics
Ali Baklouti
Hidenori Fujiwara
Jean Ludwig
Representation Theory of Solvable Lie Groups and Related Topics
Springer


Tunisian Journal of Mathematics
an international publication organized by the Tunisian Mathematical Society
2021
vol. 3
no. 4
msp