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via Carathéodory Spaces of Place Functions

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Research Advisor: Gerard Buskes

Department of Mathematics
University of Mississippi

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Riesz Spaces

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Definition

Let X be a partially ordered set. X is called a *lattice* if the supremum $(x \vee y)$ and infimum $(x \wedge y)$ exist for every pair of elements x and y in X.

Definition

Let V be a real vector space. V is an *ordered vector space* if V is partially ordered in such a way that the vector space structure and order structure are compatible. That is, for every x, y, $z \in V$ and $\lambda \geq 0$ in $\mathbb{R}$,

1. $x \leq y$ implies $x + z \leq y + z$, and
2. $x \geq 0$ implies $\lambda x \geq 0$ in V.

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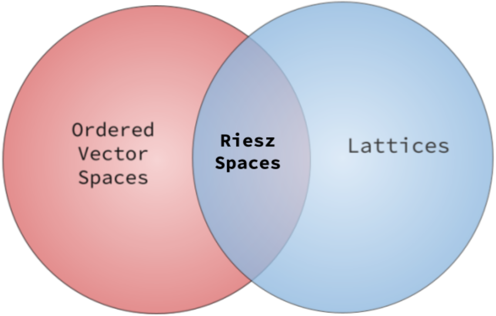
Riesz Spaces

Definition

A *Riesz space* (also called a *vector lattice*) is an ordered vector space that is also a lattice with respect to the partial ordering.



Riesz Spaces

Examples

- $V = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is a polynomial$\}$ is an ordered vector space.
- A function $p : \mathbb{R} \to \mathbb{R}$ is said to be a *piecewise polynomial* if there are $n \in \mathbb{N}$ and $t_1, \cdots, t_n \in (-\infty, \infty)$ such that $t_1 < t_2 < \cdots < t_n$ and p is a polynomial function on $(-\infty, t_1]$, $[t_n, \infty)$ and $[t_i, t_{i+1}]$ for each $i = 2, \cdots, n-1$.
- $PP(\mathbb{R})$ is a Riesz space.

Riesz Spaces

Examples

- $C(X)$ is the set of real-valued continuous functions on a topological space X, e.g., $C[0,1]$.
- $c(N)$ is the set of all convergent sequences.
- $c_0(N)$ is the set of all sequences convergent to zero.

Riesz Subspaces

Definitions

1. The linear subspace V of E is called a *Riesz subspace* of E if

$$f,\ g \in V \implies f \vee g,\ f \wedge g \in V.$$

Example: $PP([0,\infty)) \subseteq C[0,\infty)$.

2. The Riesz subspace I of E is called an *ideal* in E if

$$[f \in I,\ g \in E \text{ and } |g| \leq |f|] \implies g \in I.$$

Example: Let $a \in [0,1]$. Then $\{f \in C[0,1] : f(a) = 0\}$ is an ideal in $C[0,1]$.

3. The ideal B of E is called a *band* in E if

$$[D \subseteq B \text{ and } \sup(D) \text{ exists in } E] \implies \sup(D) \in B.$$

Example: For $a \in \mathbb{N}$, $\{f \in c(\mathbb{N}) : f(a) = 0\}$ is a band in $c(\mathbb{N})$.

Riesz Subspaces

Definitions

4. The band $[A]$ generated by the ideal A in the Riesz space E consists of all $f \in E$ satisfying

$$|f| = \sup\{u : u \in A,\ 0 \leq u \leq |f|\}.$$

5. Let $f \in E$. The *principal ideal* generated by f, denoted E_f, is the smallest ideal of E containing f. In particular,

$$E_f = \{g \in E : |g| \leq |\lambda f| \text{ for some } \lambda \in \mathbb{R}\}.$$

Bilinear Maps and the Algebraic Tensor Product

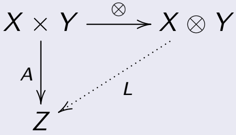
Definition

Let X, Y and Z be vector spaces. A map $T\colon X \times Y \to Z$ is *bilinear* if

$$\begin{aligned} T(x_1+x_2,y) &= T(x_1,y)+T(x_2,y) \quad (x_1,\ x_2 \in X,\ y \in Y);\\ T(x,y_1+y_2) &= T(x,y_1)+T(x,y_2) \quad (x \in X,\ y_1,\ y_2 \in Y);\\ \lambda T(x,y) &= T(\lambda x,y) = T(x,\lambda y) \quad (\lambda \in \mathbb{R},\ x \in X,\ y \in Y). \end{aligned}$$

The Universal Property

For every bilinear map $A\colon X \times Y \to Z$, there exists a unique linear map $L\colon X \otimes Y \to Z$ such that the diagram below commutes.



Riesz Bimorphisms

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Definition

Let E and F be Riesz spaces. A linear mapping $T: E \to F$ is a *Riesz homomorphism* if

$$T(x \vee y) = T(x) \vee T(y)$$

for every $x \in E$ and $y \in F$.

Definition

Let E, F, and G be Archimedean Riesz spaces. A *Riesz bimorphism* is a bilinear map $T : E \times F \to G$ such that the maps

$$z \longmapsto T(z, y) : E \to G$$
$$z \longmapsto T(x, z) : F \to G$$

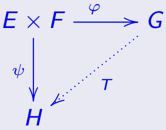
are Riesz homomorphisms for all $x \in E^+$ and all $y \in F^+$.

Fremlin Tensor Product

Theorem (Fremlin 1972)

Let E and F be Archimedean Riesz spaces. There exists an Archimedean Riesz space G and a Riesz bimorphism $\varphi\colon E \times F \to G$ with the following properties.

1. Whenever H is an Archimedean Riesz space and $\psi\colon E \times F \to H$ is a Riesz bimorphism, there is a unique Riesz homomorphism $T\colon G \to H$ such that $T \circ \varphi = \psi$.



Any G satisfying this property is the *Archimedean Riesz space* (or *Fremlin*) *tensor product* of E and F, denoted $E \bar{\otimes} F$.

Theorem (Fremlin 1972)

2 If $\psi(x, y) > 0$ in H whenever $x > 0$ in E and $y > 0$ in F, then $E\bar{\otimes}F$ may be identified with the Riesz subspace of H generated by $\psi[E \times F]$.
If $h \in E\bar{\otimes}F$, there exist finite sets I, J of $\mathbb{N}$ and $g_{ij} \in E \otimes F$ such that

$$h = \sup_{i \in I} \inf_{j \in J} \{g_{ij}\},$$

where $g_{ij} = \sum_{i=1}^{n} e_i \otimes f_i$ for some $n \in \mathbb{N}$, $e_i \in E$, and $f_i \in F$.

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Definition

Let E be an Archimedean Riesz space and let I be a nonempty set. $c_{00}(I, E)$ is the set of all maps $f : I \to E$ such that

$$S(f) = \{x \in I : f(x) \neq 0\}$$

is finite. We refer to $S(f)$ as the *support* of f. We write $c_{00}(I)$ in place of $c_{00}(I, \mathbb{R})$.

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Main Result

Theorem (Buskes, Thorn 2022)

Suppose E and F are Dedekind complete. The following are equivalent.

1. $E_x \bar{\otimes} F_y$ is Dedekind complete for every $x \in E^+$ and $y \in F^+$.
2. $[E_x$ is finite dimensional $\forall x \in E^+]$ or $[F_y$ is finite dimensional $\forall y \in F^+]$.
3. $E \cong c_{00}(I)$ for a set $I \subseteq E$ or $F \cong c_{00}(J)$ for a set $J \subseteq F$.
4. $E \bar{\otimes} F \cong c_{00}(I, F)$ for a set $I \subseteq E$ or $E \bar{\otimes} F \cong c_{00}(J, E)$ for a set $J \subseteq F$.
5. $E \bar{\otimes} F$ is Dedekind complete.

Motivation

Theorem (Fremlin 1995)

Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras. $\mathcal{A} \otimes \mathcal{B}$ is complete if and only if *either* $\mathcal{A} = \{0\}$ *or* $\mathcal{B} = \{0\}$ *or* $\mathcal{A}$ is finite and $\mathcal{B}$ is complete *or* $\mathcal{B}$ is finite and $\mathcal{A}$ is complete.

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Boolean Algebras

A lattice X is called *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all x, y, z in X.

Definition

A *Boolean algebra* is a distributive lattice with zero 0 and unit 1 having the property that every element has a complement.

Definition

A Boolean algebra is *complete* if every subset has a supremum.

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Boolean Algebra of Bands

As an intermediary between Archimedean Riesz spaces and Boolean algebras, we consider Boolean algebras of bands.

Theorem

Define $B(E) = \{B \subseteq E : B \text{ is a band}\}$.

1. $\{0\}$ *and* E *are elements of* $\mathcal{B}(E)$;
2. *intersections of bands are bands;*
3. *for any subset* D *of* E, *the disjoint complement of* D, *which is*

$$D^d = \{f \in E : |f| \wedge |g| = 0 \text{ for all } g \in D\},$$

   is an element of $\mathcal{B}(E)$.

$\mathcal{B}(E)$, *partially ordered by inclusion, is a complete Boolean algebra if* E *is Archimedean.*

Bands

Theorem (Luxemburg, Zaanen)

If the Archimedean Riesz space E has the property that any set of mutually disjoint nonzero elements is finite, then E is of finite dimension.

Lemma

Let E be a Riesz space and $f, g \in E$. Then $|f| \wedge |g| = 0$ implies $[f] \perp [g]$.

Corollary

If E is an infinite dimensional Archimedean Riesz space, then $\mathcal{B}(E)$ is not finite.

Boolean Homomorphisms

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras. A map $\chi\colon \mathcal{A} \to \mathcal{B}$ is said to be a *Boolean homomorphism* if for all $x, y \in \mathcal{A}$,

1. $\chi(x \wedge y) = \chi(x) \wedge \chi(y)$;
2. $\chi(x \oplus y) = \chi(x) \oplus \chi(y)$;
3. $\chi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

A bijective Boolean homomorphism is called a *Boolean isomorphism*. If there exists an isomorphism $\chi\colon \mathcal{A} \to \mathcal{B}$, then the Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic*.

Boolean Algebra Tensor Product

Theorem (Fremlin 1995)

1. Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of Boolean algebras. For each $i \in I$, let Z_i be the Stone space of $\mathcal{A}_i$. Set $Z = \prod_{i\in I} Z_i$, with the product topology. Then the *free product* of $\{\mathcal{A}_i\}_{i\in I}$ is the algebra of open-and-closed sets in Z, denoted $\otimes$.
2. For $i \in I$ and $a \in \mathcal{A}_i$, the set $\hat{a} \subseteq Z_i$ representing a is an open-and-closed subset of Z_i; because $z \mapsto z(i) : Z \to Z_i$ is continuous,
$$\epsilon_i(a) = \{z : z(i) \in \hat{a}\}$$
is open-and-closed, so belongs to $\mathcal{A}$. In this context, $\epsilon_i : \mathcal{A}_i \to \mathcal{A}$ is called the *canonical map*.

Boolean Algebra Tensor Product

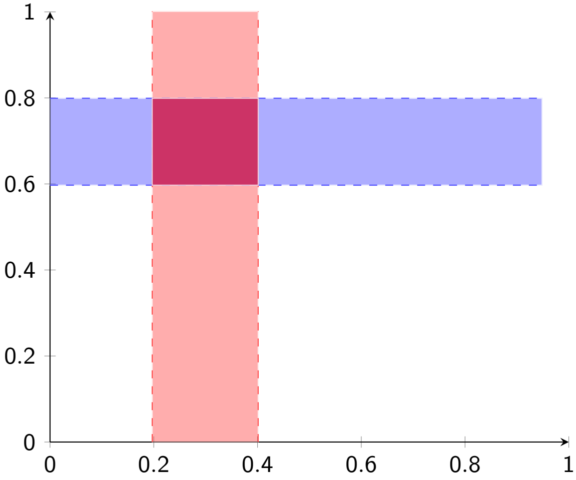
Theorem (Fremlin 1995)

Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of Boolean algebras, with free product $\mathcal{A}$.

1. The canonical map $\epsilon_i\colon \mathcal{A}_i \to \mathcal{A}$ is a Boolean homomorphism for every $i \in I$.
2. For any Boolean algebra $\mathcal{B}$ and any family $\{\varphi_i\}_{i\in I}$ such that φ_i is a Boolean homomorphism from $\mathcal{A}_i$ to $\mathcal{B}$ for every i, there is a unique Boolean homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that $\varphi_i = \varphi \circ \epsilon_i$ for each i.
3. Write C for the set of those members of $\mathcal{A}$ expressible in the form $\inf_{j\in J} \epsilon_j(a_j)$, where $J \subseteq I$ is finite and $a_j \in \mathcal{A}_j$ for every j. Then every member of $\mathcal{A}$ is expressible as the supremum of a disjoint finite subset of C.

Example: $\mathcal{A}$, $\mathcal{B}$ - collection of open and closed intervals in $\mathbb{R}$

$\epsilon_A((0.2,0.4)) \in \mathcal{A}\otimes\mathcal{B}$, $\epsilon_B((0.6,0.8)) \in \mathcal{A}\otimes\mathcal{B}$



$\epsilon_A((0.2,0.4)) \wedge \epsilon_B((0.6,0.8))$

Carathéodory Place Functions

Definition

Let E be a Riesz space and $e \in E^+$. Then $x \in E^+$ is said to be a *component* of e whenever $x \wedge (e - x) = 0$. $\mathcal{C}(e)$ is the set of all component of e.

Theorem (Buskes, de Pagter, van Rooij 2008)

Let $\mathcal{A}$ be a Boolean algebra. There exists an Archimedean Riesz space E with an order unit e with the following properties.

1. There exists a Boolean isomorphism $\chi \colon \mathcal{A} \to \mathcal{C}(e)$.
2. E is the linear span of $\mathcal{C}(e)$.

(E, χ) is unique up to isomorphism. It is denoted by $\mathcal{C}(\mathcal{A})$ and is called the *space of place functions on $\mathcal{A}$* in the sense of Carathéodory.

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Complete Boolean Algebra

Theorem

Let $\mathcal{A}$ be a Boolean algebra. $\mathcal{A}$ is complete if and only if $\mathcal{C}(\mathcal{A})$ is Dedekind complete.

Main Boolean Algebra Result

Theorem

$C(\mathcal{A})\bar{\otimes}C(\mathcal{B})$ and $C(\mathcal{A}\otimes\mathcal{B})$ are Riesz isomorphic.

Sketch of Proof: For $f\in\mathcal{C}(\mathcal{A})$ and $g\in\mathcal{C}(\mathcal{B})$, there exist n, $m\in\mathbb{N}$, pairwise disjoint $x_i\in\mathcal{C}(a)$, pairwise disjoint $u_j\in\mathcal{C}(b)$, and nonzero λ_i, $\gamma_j\in\mathbb{R}\in\mathbb{R}$ such that $f=\sum_{i=1}^{n}\lambda_i\chi_A(x_i)$ and $g=\sum_{j=1}^{m}\gamma_j\chi_B(u_j)$. Define $\psi\colon\mathcal{C}(\mathcal{A})\times\mathcal{C}(\mathcal{B})\to\mathcal{C}(\mathcal{A}\otimes\mathcal{B})$ by

$$\begin{aligned}\psi(f,g)=&\psi\left(\sum_{i=1}^{n}\lambda_i\chi_A(x_i),\sum_{j=1}^{m}\gamma_j\chi_B(u_j)\right)\\=&\sum_{i=1}^{n}\sum_{j=1}^{m}(\lambda_i\gamma_j)\hat{\chi}(\epsilon_A(x_i)\wedge\epsilon_B(u_j)).\end{aligned}$$

ψ is a Riesz bimorphism,so there exists a unique Riesz homomorphism $T\colon C(\mathcal{A})\bar{\otimes}C(\mathcal{B})\to C(\mathcal{A}\otimes\mathcal{B})$ such that $\psi=T\circ\otimes$.

$T\colon C(\mathcal{A})\bar{\otimes}C(\mathcal{B}) \to C(\mathcal{A}\otimes\mathcal{B})$ is onto.

Let $h \in C(\mathcal{A}\otimes\mathcal{B})$.Then $h = \sum_{i=1}^{n} \lambda_i \hat{\chi}(e_i)$ for some pairwise disjoint $e_i \in \mathcal{A}\otimes\mathcal{B}$, $n \in \mathbb{N}$, and nonzero $\lambda_i \in \mathbb{R}$. Fix $i \in \{1, \cdots, n\}$. By Fremlin's properties of $\mathcal{A}\otimes\mathcal{B}$, there exists a finite disjoint subset $\{\epsilon_A(a_k)\wedge\epsilon_B(b_k)\}_{k=1}^{m}$ $(m \in \mathbb{N})$ of $\mathcal{A}\otimes\mathcal{B}$ such that

$$e_i = \bigvee_{k=1}^{m} \epsilon_A(a_k)\wedge\epsilon_B(b_k).$$

Then it follows from the definition of ψ that

$$\begin{aligned}
\hat{\chi}(e_i) =& \hat{\chi}\left(\bigvee_{k=1}^{m} \epsilon_A(a_k)\wedge\epsilon_B(b_k)\right) \\
=& \bigvee_{k=1}^{m} \hat{\chi}(\epsilon_A(a_k)\wedge\epsilon_B(b_k)) \\
=& \bigvee_{k=1}^{m} \psi(\chi_A(a_k), \chi_B(b_k)) \\
=& \bigvee_{k=1}^{m} T\circ\otimes(\chi_A(a_k), \chi_B(b_k)).
\end{aligned}$$

Since T preserves finite suprema, $\hat{\chi}(e_i)$ is in the image of T for every i.It follows from the linearity of T that h is in the image of T.

$T\colon C(\mathcal{A})\bar{\otimes}C(\mathcal{B}) \to C(\mathcal{A}\otimes\mathcal{B})$ is one-to-one

Suppose $f \in C(\mathcal{A}) \otimes C(\mathcal{B})$, the algebraic tensor product of $C(\mathcal{A})$ and $C(\mathcal{B})$, such that f is nonzero. Then for some $n \in \mathbb{N}$, nonzero $\lambda_k \in \mathbb{R}$, and nontrivial $x_k \in \mathcal{A}$, $u_k \in \mathcal{B}$ such that

$$f = \sum_{k=1}^{n} \lambda_k \chi_A(x_k) \otimes \chi_B(u_k).$$

Since ϵ_A, ϵ_B, and $\hat{\chi}$ are injective Boolean isomorphisms,

$$\begin{aligned} T(f) &= T\left(\sum_{k=1}^{n} \lambda_k \chi_A(x_k) \otimes \chi_B(u_k)\right) \\ &= \sum_{k=1}^{n} \lambda_k \psi\left(\chi_A(x_k), \chi_B(u_k)\right) \\ &= \sum_{k=1}^{n} \lambda_k \hat{\chi}(\epsilon_A(x_k) \wedge \epsilon_B(u_j)) \\ &\neq 0. \end{aligned}$$

$T\colon C(\mathcal{A})\bar{\otimes}C(\mathcal{B}) \to C(\mathcal{A}\otimes\mathcal{B})$ is one-to-one

Let $g \in C(\mathcal{A})\bar{\otimes}C(\mathcal{B})$ such that $g \neq 0$.
By Theorem 2.2 of [1], for all $\delta > 0$ there exists $f \in \mathcal{C}(\mathcal{A})^+ \otimes \mathcal{C}(\mathcal{B})^+$ such that

$$0 \leq |g| - f \leq \delta\hat{\chi}(1_{\mathcal{A}\otimes\mathcal{B}}).$$

Since $C(\mathcal{A})\bar{\otimes}C(\mathcal{B})$ is Archimedean, choose $\delta > 0$ such that $|g| \wedge \delta\hat{\chi}(1_{\mathcal{A}\otimes\mathcal{B}}) \neq |g|$. Then f is nonzero.
We have shown that $T(f) \neq 0$ when $0 \neq f \in \mathcal{C}(\mathcal{A}) \otimes \mathcal{C}(\mathcal{B})$.
Since T is a Riesz homomorphism, $0 < T(f) \leq |T(g)|$. Therefore, $T(g) \neq 0$, and T is a Riesz isomorphism.

Finally, $C(\mathcal{A})\bar{\otimes}C(\mathcal{B})$ is Riesz isomorphic to $C(\mathcal{A}\otimes\mathcal{B})$.

Applications

Theorem (Fremlin 1995)

Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras. $\mathcal{A} \otimes \mathcal{B}$ is complete if and only if *either* $\mathcal{A} = \{0\}$ *or* $\mathcal{B} = \{0\}$ *or* $\mathcal{A}$ is finite and $\mathcal{B}$ is complete *or* $\mathcal{B}$ is finite and $\mathcal{A}$ is complete.

Proof.

($\Longrightarrow$) It follows that $\mathcal{C}(\mathcal{A} \otimes \mathcal{B}) \cong \mathcal{C}(\mathcal{A})\bar{\otimes}\mathcal{C}(\mathcal{B})$ is Dedekind complete. Then $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$ are Dedekind complete, and so $\mathcal{A}$ and $\mathcal{B}$ are complete. It remains to show that one of the Boolean algebras is finite.

By our main result on Dedekind completeness, the Dedekind completeness of $\mathcal{C}(\mathcal{A})\bar{\otimes}\mathcal{C}(\mathcal{B})$ implies that $\mathcal{C}(\mathcal{A}) \cong c_{00}(I)$ for a set $I \subseteq \mathcal{C}(\mathcal{A})$ or $\mathcal{C}(\mathcal{B}) \cong c_{00}(J)$ for a set $J \subseteq \mathcal{C}(\mathcal{B})$. Since each Carathéodory space of place functions contains a unit, $\mathcal{C}(\mathcal{A})$ or $\mathcal{C}(\mathcal{B})$ is finite dimensional. Thus, $\mathcal{A}$ is finite or $\mathcal{B}$ is finite.

($\Longleftarrow$) The sufficiency is proven similarly via our main result on the Dedekind completeness of the tensor product. □

Comparison of Results

Theorem (Fremlin 1995)

Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras. $\mathcal{A} \otimes \mathcal{B}$ is complete if and only if *either* $\mathcal{A} = \{0\}$ *or* $\mathcal{B} = \{0\}$ *or* $\mathcal{A}$ is finite and $\mathcal{B}$ is complete *or* $\mathcal{B}$ is finite and $\mathcal{A}$ is complete.

Theorem (Buskes, Thorn 2022)

Suppose E and F are Dedekind complete. The following are equivalent.

1. $E_x \bar{\otimes} F_y$ is Dedekind complete for every $x \in E^+$ and $y \in F^+$.
2. [E_x is finite dimensional $\forall x \in E^+$] or [F_y is finite dimensional $\forall y \in F^+$].
3. $E \cong c_{00}(I)$ for a set $I \subseteq E$ or $F \cong c_{00}(J)$ for a set $J \subseteq F$.
4. $E \bar{\otimes} F \cong c_{00}(I, F)$ for a set $I \subseteq E$ or $E \bar{\otimes} F \cong c_{00}(J, E)$ for a set $J \subseteq F$.
5. $E \bar{\otimes} F$ is Dedekind complete.

Applications

Corollary

Let E and F be infinite dimensional Archimedean Riesz spaces. Then $\mathcal{B}(E) \otimes \mathcal{B}(F)$ is not Boolean isomorphic to $\mathcal{B}(E\bar{\otimes}F)$.

Proof.

Since E and F are infinite dimensional, neither $\mathcal{B}(E)$ nor $\mathcal{B}(F)$ is finite. Then $\mathcal{B}(E) \otimes \mathcal{B}(F)$ is not complete by the previous theorem. However, the Boolean algebra of bands is complete for any Archimedean Riesz space, so $\mathcal{B}(E\bar{\otimes}F)$ is complete. □

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Thank you!

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