Stochastique matrices and polytopes

Dorsaf Cherif
University Tunis El Manar

December 7, 2022

Affine subspaces

- We work in $\mathbb{R}^d$.
- A non empty affine subspace is a translate of a linear subspace.
- The dimension of an affine subspace is the dimension of the corresponding linear vector subspace.
- Affine maps are in the form : $x \longmapsto Ax + x_0$.

Polytopes

The following elementary notions on polytopes are from [2].

Definition

A polytope is the convex hull of a finite set of points in some $\mathbb{R}^d$.

- The dimension of a polytope is the dimension of it's affine hull.
- Two dimentional polytopes are called polygones.

Definition

A half space is a set $P \subset \mathbb{R}^d$ presented in the form $P = \{X \in \mathbb{R}^d : AX \leq Z\}$ for some $A \in M_{n,d}(\mathbb{R})$ and $Z \in \mathbb{R}^n$.
A polyhedron is an intersection of closed half spaces.

Theorem

$P \subset \mathbb{R}^d$ is a polytope if and only if P is a bounded polyhedron.

Faces

Definition

$F \subset \mathbb{R}^d$ is called a face of a polytope P if there exists an hyperplane $H = \{X \in \mathbb{R}^d :^t CX = c_0\}$, for some $C \in \mathbb{R}^d$ and $c_0 \in \mathbf{R}$, such that $F = P \cap H$ where ${}^tCX \leq c_0$ is a valid inequality for P.

Example

$OX \leq 0$ is a valid inequality for P then P itself is a face of P.
$OX \leq 1$ gives that $\emptyset$ is a face of P.

Faces of dimension 0 are called vertices, of dimension 1 edges and of dimension $dim(P) - 1$ facets.

Vertices

Let us denote $vert(P)$ the set of vertices of P.

Proposition

Consider P a polytope, then $P = conv(vert(P))$.

Proposition

Consider P a polytope and $V = vert(P)$. Let F be a face of P. Then:

1. *F is a polytope and $vert(F) = F \cap V$.*
2. *Every intersection of faces is a face.*
3. *The faces of F are exactly the faces of P contained in F.*
4. *$F = P \cap aff(F)$.*

Characterization of the nonnegative idempotent matrices:

Theorem (P. Flor)

Let A be a nonnegative idempotent matrix of rank k. Then, there exists a permutation matrix P such that

$$PA^tP = \left(\begin{array}{c|c|c|c} J & JU & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline VJ & VJU & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

V and U non negative matrices with the appropriate size and

$$J = \left(\begin{array}{c|c|c|c} J_1 & 0 & \cdots & 0 \\ \hline 0 & J_2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & \cdots & 0 & J_r \end{array}\right)$$ *each J_k is rank one.*

Steps of the proof

The following proof is from [3]. For a proof that does not use polytopes one can see [4]

Proof.

Assume first that A has no zero rows and no zero column. Consider:

- $e_1, .., e_n$ the canonical bases of $\mathbb{R}^n$.
- $C_1, .., C_n$ the columns of A.
- K the polytope generated by $C_1, ..C_n$.
- $E_1, .., E_k$ the edges of K.
- $M_j = \{e_i, i \in [1, n] : C_i \in E_j\}$. $S_j = vect(M_j)$.

Then $AS_j = E_j$. The restriction of A to S_j has rank one. By rearranging the cordinate in such a way that the unit vectors belonging to each S_j are regroupped together. We obtain the matrix J of the theorem. □

Proof.

For the general case there exists a permutation matrix P_1 such that

$$P_1 A^t P_1 = \left(\begin{array}{c|c|c|c} X & B & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline C & D & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

X and B have no zero rows in common and X and C have no zero columns in common. A is idempotent then $X^2 = X$, $XB = B$ $CX = C$ and $CB = D$. Then X has no zero columns and rows. So we apply the first case to X. There exists a permutation matrix P_2 such that $J = P_2 X^t P_2$ is in the form we want. Then setting $U = P_2 B$ and $V = C^t P$. Taking $P = P_3 P_1$ with $P_3 = \left(\begin{array}{c|c} P_2 & 0 \\ \hline 0 & I \end{array}\right)$. $PA^t P$ has the form given by the theorem. □

Stochastique matrices

Theorem

Let A be an idempotent stochastic matrix of rank k. Then, A is an idempotent if and only if there exists a permutation matrix P such that

$$PA^tP = \left(\begin{array}{c|c} J & 0 \\ \hline VJ & 0 \end{array}\right)$$

V arbitrary positive and

$$J = \left(\begin{array}{c|c|c|c} J_1 & 0 & \cdots & 0 \\ \hline 0 & J_2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & \cdots & 0 & J_r \end{array}\right)$$

each J_k is rank one.

Doubly stochastique matrices.

Theorem

Let A be an idempotent doubly stochastic matrix of rank k. Then, A is an idempotent if and only if there exists a permutation matrix P such that

$$PA^tP = \left(\begin{array}{c|c|c|c} J_1 & 0 & \cdots & 0 \\ \hline 0 & J_2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & \cdots & 0 & J_r \end{array}\right)$$

each J_k is rank one where, if J_i is $n_i \times n_i$ then each entry of J_i is $1/n_i$.

Example

Consider the idempotent stochastique of rank 2 $A = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \end{pmatrix}$. We take $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. We obtain $J = PA^tP = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Remark

The J_i in the matrix J are of rank one. So there exists X_i and Y_i in $\mathbb{R}^{d_i}$ such that $J_i = X_i^t Y_i$ and $1 =^t Y_i X_i$.

We denote by

- St_n the semigroup of stochastic matrices of order n.
- D_n the semigroup of doubly stochastic matrices of order n.

Theorem

The set St_n is a convex polytope of dimension $n(n-1)$ whose vertices are all the stochastic matrices having 0 or 1 in any of their entries, n^n vertices in total.

Example

The 2-dimensional compact affine semigroup St_2 has the four vertices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Birkhoff's theorem

Theorem (Birkhoff)

The set D_n is a convex polytope of dimension $(n-1)^2$ whose vertices constitute the permutation matrices of order n, that is, every doubly stochastique matrice is a convex combination of permutation matrices.

We associate to a doubly stochastic matrix a bipartite graph as follows. We represent each row and each column with a vertex and we connect the vertex representing row i with the vertex representing row j if the entry $x_{i,j}$ in the matrix is not zero.
To prove this theorem we need the following lemma. For details see [5].

Perfect matching

Definition

A perfect matching in a graph G is a matching in which every vertex of G appears exactly once, that is, a matching of size exactly $n/2$.

Lemma

The associated graph of any doubly stochastic matrix has a perfect matching.

Proof of Birkhoff's

Proof.

We proceed by induction on the number of nonzero entries in the matrix. Let M_0 be a doubly stochastic matrix. By the key lemma, the associated graph has a perfect matching. Underline the entries associated to the edges in the matching. Thus we underline exactly one element in each row and each column. Let α_0 be the minimum of the underlined entries. Let P_0 be the permutation matrix that has a 1 exactly at the position of the underlined elements. If $\alpha_0 = 1$ then all underlined entries are 1, and $M_0 = P_0$ is a permutation matrix. If $\alpha_0 < 1$ then the matrix $M_0 - \alpha_0 P_0$ has non-negative entries, and the sum of the entries in any row or any column is $1 - \alpha_0$. Dividing each entry by $(1 - \alpha_0)$ in $M_0 - \alpha_0 P_0$ gives a doubly stochastic matrix M_1 . Thus we may write $M_0 = \alpha_0 P_0 + (1 - \alpha_0) M_1$ where M_1 is not only doubly stochastic, but has less non-zero entries than M_0 . We conclude by induction . □

Maximal monoid

Definition

Let S be a semigroup and e be an idempotent of S. Then, $eSe = \{x \in S | ex = xe = x\}$ is called the maximal monoid of S relative to e.

For the proof of the following theorem see [1].

Theorem

Let A be an idempotent of rank k in St_n. Then, ASt_nA is affinely isomorphic to St_k. The maximal monoid ASt_nA is a convex polytope of dimension $k(k-1)$ with k^k vertices.

References

A geometric description of the maximal monoidsof some matrix semigroups Raul E. Gonzalez-Torres

Lectures on Polytopes. Gunter M.Ziegler , New York (1995)

Non negative matrices in the mathematical sciences. A.Berman, R.Plemmons.

Constructing the maximal monoids in the semigroupsN_n, S_n and Ω_n. D.J.HARTFIEL, C.J.MAXSON

Gabor Hetyei. Birkhoff's theorem. 2007.

Thank you for your attention.