Tensor products of Archimedean partially ordered vector spaces

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Overview

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Introduction

(Fremlin, 1972)

- construction of the Riesz tensor product
- representation as the space of continuous functions : $C(X) \otimes C(Y)$ linear subspace of $C(X \times Y)$

(Grobler and Labuschagne, 1988)

- construction of the Fremlin tensor product by means of Dedekind completions
- construction of the tensor product of directed Archimedean POVS with the Riesz decomposition property
- the relative uniform closure of the projective cone is again a cone

Gaans and Kalauch, 2010

- showed ru-closure of the projective cone in $X \otimes Y$ is again a cone by using Riesz completions of Archimedean POVS X and Y and their Fremlin tensor product

Algebraic tensor product and the projective cone

Tensor product (algebraic)

Let X and Y be real vector spaces. The tensor product of X and Y is a pair (T, τ) satisfiying the following:

- T is a vector space and $\tau : \ X \times Y \to T$ is a bilinear map
- if S is a vector space and $\sigma : \ X \times Y \to S$ is a bilinear map, then there is a unique linear map $\sigma^\star : \ T \to S$ such that $\sigma(x, y) = \sigma^\star(\tau(x, y))$ for all $x \in X$ and $y \in Y$.

There exists an essentially unique tensor product (T, τ) of X and Y and we denote it by $X \otimes Y$. For $x \in X$ and $y \in Y$,

$$x \otimes y = \tau(x, y)$$

Algebraic tensor product and the projective cone

Cone

- Let X be a real vector space. A nonempty set $K \subseteq X$ is called a wedge in X if $x, y \in K$, $\lambda, \mu \in [0, \infty)$ imply $\lambda x + \mu y \in K$.
- If K is a wedge in X with the additional property $K \cap (-K) = 0$ then K is called a cone in X.

Partial order

$$y \geq x \Longleftrightarrow y - x \in K$$

POVS

A vector space X with a given cone K equipped with the vector space order is called a partially ordered vector space.

Algebraic tensor product and the projective cone

(X, K) is called Archimedean if for every $x, y \in X$ with $nx \leq y$ for all $n \in \mathbf{N}$, one has $x \leq 0$.

A set $M \subseteq X$ is called directed if for every $x, y \in M$, there is an element $z \in M$ such that $z \geq x$ and $z \geq y$.

We say that X has the Riesz decomposition property if for every $y, x_1, x_2 \in K$ with $y \leq x_1 + x_2$, there exist $y_1, y_2 \in K$ such that $y = y_1 + y_2$ and $y_1 \leq x_1$, $y_2 \leq x_2$.

We say that K is generating X if $K = K - K$

Algebraic tensor product and the projective cone

Projective cone

We define the projective cone in the algebraic tensor product $T = X \otimes Y$ as

$$K_T := \{\sum_{i=1}^{n} \alpha_i x_i \otimes y_i : \ x_i \in K_X, \ y_i \in K_Y, \ \alpha_i \in \mathbf{R}^+, \ n \in \mathbf{N}\}$$

Theorem (Gaans, Kalauch, 2010)

(T, K_T) is a partially ordered vector space. If X and Y are directed partially ordered vector spaces, then K_T is generating in T .

Riesz completions and Fremlin's tensor product as Riesz completion

We denote for a subset $M \subseteq X$ the set of all upper bounds by

$$M^u = \{x \in X : \ x \geq m \ \text{ for all } m \in M\}$$

pre-Riesz space

A POVS X is called

- a pre-Riesz space if for every $x, y, z \in X$ the inclusion $\{x+y, x+z\}^u \subseteq \{y, z\}^u$ implies $x \in K$
- a Riesz space if the ordering is a lattice ordering.

$$\text{pre-Riesz} \implies \text{directed}$$

$$\text{directed Archimedean POVS} \implies \text{pre-Riesz}$$

$$\text{Riesz} \implies \text{pre-Riesz}$$

Riesz completions and Fremlin's tensor product as Riesz completion

Examples

- If $K = \{(x_1, x_2)^T;\ x_1 \geq 0, x_2 \geq 0\}$, then K induces the standard order on $\mathbf{R}^2$, so that $(\mathbf{R}^2, K)$ is a Riesz space and hence a pre-Riesz space.
- $X = \mathbf{R}^2$ provided with the partial ordering defined by

$$(a, b) \leq (c, d) \Longleftrightarrow (a < c \text{ and } b < d) \text{ or } (a = c \text{ and } b = d)$$

is a directed POVS which is not pre-Riesz. Take $x = (1, 0)$ and $A = \{(-1, 0), (0, 0)\}$ as $(x + A)^u \subseteq A^u$. But $x \ngeq 0$.

Riesz completions and Fremlin's tensor product as Riesz completion

We say that a subspace X of a Riesz space Y generates Y as a Riesz space if for every $y \in Y$ there exist $a_1, ..., a_m, b_1, ..., b_n \in X$ such that

$$y = \bigvee_{i=1}^{m} a_i - \bigvee_{i=1}^{n} b_i$$

Order denseness

A linear subspace D of a POVS X is called order dense in X if for every $x \in X$ we have $x = \inf\{y \in D : \ y \geq x\}$.

Let X and Y be directed POVS. The linear map $i : \ X \longrightarrow Y$ is

- a positive map if $x \leq y \Longrightarrow i(x) \leq i(y)$
- a bipositive map if $x \leq y \Longleftrightarrow i(x) \leq i(y)$, for all $x, y \in X$.

Riesz completions and Fremlin's tensor product as Riesz completion

Van Haandel, 1993

These statements are equivalent:

1. X is pre-Riesz.
2. There exist a Riesz space Y and a bipositive linear map $i : X \longrightarrow Y$ such that $i(X)$ is order dense in Y.
3. There exist a Riesz space Y and a bipositive linear map $i : X \longrightarrow Y$ such that $i(X)$ is order dense in Y and generates Y as a Riesz space.

A pair (Y, i) as in 3 is called a Riesz completion of X. We will denote it as X^ρ.

The Riesz completion is unique up to Riesz isomorphisms. Every directed partially ordered vector space has a Riesz completion.

Riesz completions and Fremlin's tensor product as Riesz completion

Van Haandel, 1993

Let X and Y be directed POVS. A linear map $h: X \longrightarrow Y$ is called a Riesz* homomorphism if for any $a, b \in X$ and for every lower bound x of $\{a, b\}^u$ in X one has that $h(x)$ is a lower bound of $\{h(a), h(b)\}^u$ in Y

- If X, Y are Riesz spaces, then $h: X \longrightarrow Y$ is Riesz* homomorphism $\Longleftrightarrow$ h is Riesz homomorphism.
- If X, Y are pre-Riesz spaces, then $h: X \longrightarrow Y$ is Riesz* homomorphism $\Longleftrightarrow$ h is the restriction of a Riesz homomorphism from X^ρ to Y^ρ.

Fremlin's tensor product

Theorem (Fremlin,1972)

Let E and F be Archimedean Riesz spaces. Then there is an Archimedean Riesz space G and a Riesz bimorphism $\varphi\colon E \times F \to G$ such that

(i) whenever H is an Archimedean Riesz space and $\psi\colon E \times F \to H$ is a Riesz bimorphism, there is a unique Riesz homomorphism $T\colon G \to H$ such that $T\varphi = \psi$;

(ii) φ induces an embedding $\hat{\varphi}\colon E \otimes F \to G$;

(iii) (ru-D) $\hat{\varphi}[E \otimes F]$ is dense in G in the sense that for every $w \in G$, there exist $x_0 \in E$ and $y_0 \in F$ such that for every $\epsilon > 0$, there is an element $v \in \hat{\varphi}[E \otimes F]$ such that $|w - v| \leq \epsilon\hat{\varphi}(x_0 \otimes y_0)$;

(iv) if $w > 0$ in G, then there exist $x \in E^+$ and $y \in F^+$ such that $0 < \hat{\varphi}(x \otimes y) \leq w$.

This unique Archimedean Riesz space G is called the Fremlin tensor product of E and F and is denoted by $E\bar{\otimes}F$.

Riesz completions and Fremlin's tensor product as Riesz completion

Gaans, Kalauch, 2010

Theorem

Let E and F be Archimedean Riesz spaces, let $E\bar{\otimes}F$ be the Fremlin tensor product of E and F, and let $E \otimes F$ be the linear subspace generated by all $x \otimes y$, $x \in E$, $y \in F$, endowed with the induced order. Then $E \otimes F$ is a pre-Riesz space and $E\bar{\otimes}F$ is its Riesz completion. Moreover, the inclusion map $\hat{\varphi}\colon E \otimes F \to E\bar{\otimes}F$ is a Riesz homomorphism.*

Construction of the cone in the tensor product of directed Archimedean POVS

Archimedean tensor cone(Grobler, Labuschagne, 1988)

A cone K in $X \otimes Y$ is called an Archimedean tensor cone if $K_T \subseteq K$ and the following universal mapping property is satisfied: For every directed Archimedean POVS (S, K_S) and every positive bilinear map $\sigma : X \times Y \longrightarrow S$ the induced linear map $\sigma^* : (X \otimes Y, K) \longrightarrow (S, K_S)$ is positive.

Lemma (Gaans,Kalauch)

Let X, Y, U and V be vector spaces and let $\rho_X : X \longrightarrow U$ and $\rho_Y : Y \longrightarrow V$ be linear injections. Let $\rho(x, y) := \rho_X(x) \otimes \rho_Y(y), x \in X, y \in Y$. Then the unique linear map $\rho^* : X \otimes Y \longrightarrow U \otimes V$ satisfying $\rho^*(x \otimes y) = \rho(x, y)$ for all $x \in X$ and $y \in Y$ is injective.

Construction of the cone in the tensor product of directed Archimedean POVS

Let (X^ρ, ρ_X) and (Y^ρ, ρ_Y) be the Riesz completions of X and Y. By the theorem of Fremlin, there exists a Riesz bimorphism

$$\phi_F : \; X^\rho \times Y^\rho \longrightarrow X^\rho \bar{\otimes} Y^\rho$$

and there is a linear injection

$$h_F : \; X^\rho \otimes Y^\rho \longrightarrow X^\rho \bar{\otimes} Y^\rho$$

such that $h_F(u \otimes v) = \phi_F(u, v)$ for all $u \in X^\rho$ and $v \in Y^\rho$. Define

$$\rho : \; X \times Y \longrightarrow X^\rho \otimes Y^\rho$$

$$\rho(x, y) := \rho_X(x) \otimes \rho_Y(y), \; x \in X, y \in Y$$

Construction of the cone in the tensor product of directed Archimedean POVS

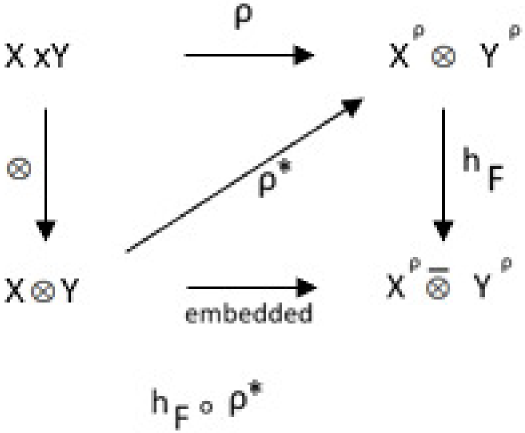
ρ is bilinear, it induces a unique linear map

$$\rho^* : \ X \otimes Y \longrightarrow X^\rho \otimes Y^\rho$$

such that $\rho(x, y) = \rho^*(x \otimes y)$. By the previous lemma, ρ^* is injective, so $X \otimes Y$ is embedded into $X^\rho \bar{\otimes} Y^\rho$ by the injective linear map $h_F \circ \rho^*$. So the order in $X^\rho \bar{\otimes} Y^\rho$ induces an order on $X \otimes Y$.

$$K_F := \ \{w \in X \otimes Y : \ h_F(\rho^*(w)) \in (X^\rho \bar{\otimes} Y^\rho)^+\}$$

Construction of the Archimedean cone



relative uniform topology

A sequence $(s_n)_n$ in a directed POVS S is said to converge relatively uniformly to an $s \in S$, denoted by $s_n \xrightarrow{(ru)} s$, if there exist an $a \in K_S$ and a sequence $(\lambda_n)_n$ in $\mathbf{R}^+$ such that $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $-\lambda_n a \leq s_n - s \leq \lambda_n a$ for all n.

Lemma (Gaans, Kalauch)

- K_F is a cone in $X \otimes Y$, $K_T \subseteq K_F$, and $(X \otimes Y, K_F)$ is Archimedean.
- K_F is ru-closed in $(X \otimes Y, K_T)$.

Theorem (Gaans, Kalauch)

For a cone K in $X \otimes Y$ following are equivalent:

1. K is the Archimedean tensor cone
2. Let (S, K_S) be a directed POVS, $\phi : \ X \otimes Y \longrightarrow S$ be a linear map such that $\phi(w) \in K_S$ for all $w \in K_T$. Then $\phi(w) \in K_S$ for all $w \in K$.
3. K is the smallest Archimedean cone in $X \otimes Y$ with $K_T \subseteq K$
4. $K = \bar{K_T}$, where $\bar{K_T}$ is the ru-closure of K_T in $(X \otimes Y, K_T)$

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THANK YOU...