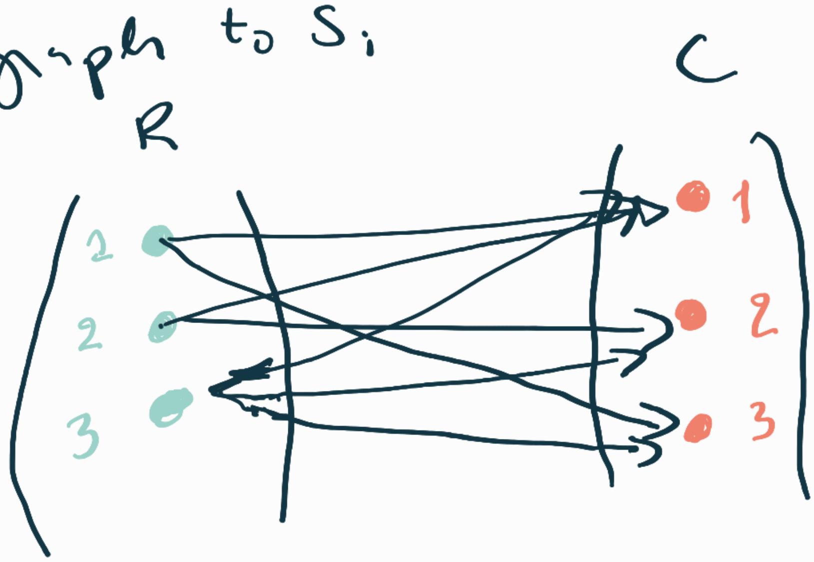
Proof of Birkoff's theorem :

Example : $S = \begin{bmatrix} \frac{3}{10} & 0 & \frac{7}{10} \\ \frac{1}{10} & \frac{9}{10} & 0 \\ \frac{6}{10} & \frac{1}{10} & \frac{3}{10} \end{bmatrix} \in D_3$

we associate the following bipartite graph to S;

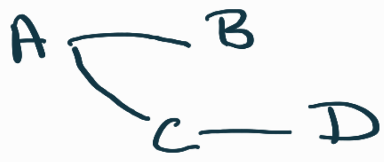


Each row and each column is represented by a vertex and we connect the vertex representing row i to the column j if $s_{ij} \neq 0$

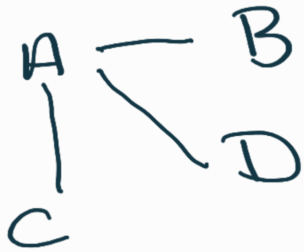
$(1, 3), (2, 1), (3, 2)$

is a perfect matching for S.

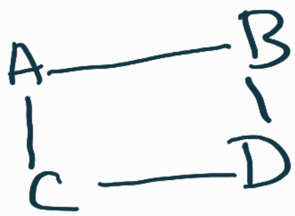
Remark:



A — B
C — D   is a perfect matching



has no perfect matching



A — B   or   A — C
C — D   B — D

Back to the example

$(1,3)$, $(2,1)$ and $(3,2)$ is a perfect matching.

This results in the following permutation matrix:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

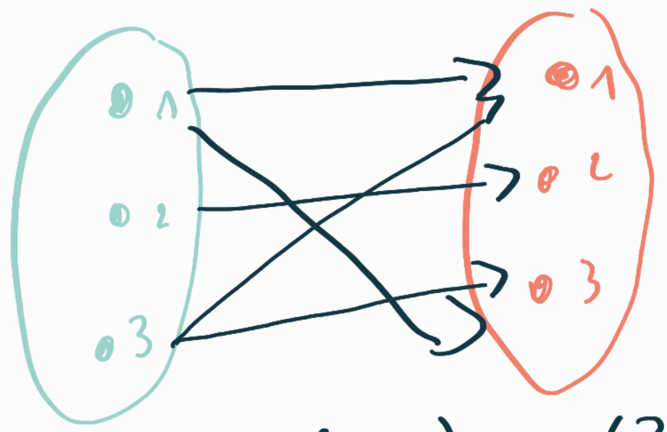
$$S_0 = S = \begin{bmatrix} \frac{3}{10} & 0 & \boxed{\frac{7}{10}} \\ \boxed{\frac{1}{10}} & \frac{9}{10} & 0 \\ \frac{6}{10} & \boxed{\frac{1}{10}} & \frac{3}{10} \end{bmatrix}$$

$\lambda_0 = \frac{1}{10}$, $P = R_0$

$$S_1 = \frac{1}{1 - \frac{1}{10}} \left(S_0 - \frac{1}{10} R_0 \right)$$

$$= \begin{bmatrix} \frac{3}{9} & 0 & \frac{6}{9} \\ 0 & 1 & 0 \\ \frac{6}{10} & 0 & \frac{3}{9} \end{bmatrix}$$

$S_1 \in D_3$



$(1,1)$, $(2,2)$, $(3,3)$.

$P_1 = I_3$, $\lambda_1 = \frac{3}{9}$

$$S_2 = \frac{1}{\left(1-\frac{3}{9}\right)}\left(S_1 - \frac{3}{9}P_1\right)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = S_2$$

$S_2 = P_2$ a permutation matrix

So we are done.

$$S_0 = \frac{1}{10}P_0 + \frac{3}{10}P_1 + \frac{6}{10}P_2$$

✓