LECTURES ON THE ENTROPY THEORY OF MEASURE PRESERVING SYSETEM BY ROKHLIN (PART 1)

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Outline

A map from a measure space to another is said to be:

- homomorphism: if the inverse image of a measurable set is measurable and has the same measure.
- isomorphism: if it is a one-to-one homomorphism and the inverse map is also a homomorphism.

If the spaces coincide

- homomorhism = endomorphism .
- isomorphism = automorphism .

Two homomorphisms will be identified mod 0 if they differ on their domains and ranges on sets of measure zero.

- A basis of a space M: a countable system $\{B_a\}_{a \in A}$ of measurable sets satisfying:
  - $\forall$ X measurable set, $\exists Y$ a set in the Borel set generated by $\{B_a\}_{a \in A}$ such that $X \subset Y$ and $\mu(Y - X) = 0$.
  - $\forall$ $x, y \in M$, $\exists a \in A$ such that either $x \in B_a$, $y \notin B_a$, or $x \notin B_a$, $y \in B_a$.
- seperable space: space with a complete normalized measure.
- Lebesgue space: seperable space complete mod 0 with respect to one basis.

Let M be a Lebesgue space.

- Partition of M: Collection of non-empty disjoint sets covering M.
- ξ-sets: subsets of M that are sums of elements of a partition ξ.
- basis of ξ: a countable system $\{B_a\}_{a \in A}$ of measurable ξ-sets satisfying for any $C, C' \in \xi$, there exists $a \in A$ such that $C \subset B_a, C' \not\subset B_a$ or $C \not\subset B_a, C' \subset B_a$.
- partition + basis = measurable partition

Isometric operators

Let H be a separable unitary space.

- Isometric operator in H: an isomorphism transformation of H onto a subspace.
- An isometric operator U acting on H is said to be
  - unitary if $UH = H$.
  - semi-unitary if UH is a proper subspace of H.
- A subspace G of H is said to be
  - invariant under U if $UG \subset G$.
  - completely invariant under U if $UG = G$.

Adjoint operator

Every endomorphism T of a space M has an adjoint operator U_T acting in $L^2(M)$ and defined by

$$U_T f(x) = f(Tx), \quad f \in L^2(M), \quad x \in M.$$

- U_T is an isometric operator.

$$||U_T f||_2^2 = \int |U_T f|^2 d\mu = \int |f(Tx)|^2 d\mu = ||f||_2^2.$$

- If T is an automorphism, U_T is unitary.
- otherwise U_T is semi-unitary.

Invariance under an endomorphism T

A measurable partition ξ of M is said to be

- invariant under T if $T^{-1}\xi \leq \xi$.
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Now, if T is an automorphism,

- $\xi_T = \bigvee_{-\infty}^{\infty} T^k\xi$ is the coarsest completely invariant measurable partition finer than ξ.
- ξ is invariant $\iff \xi = \xi_T$.

Ergodicity

Definition

An endomorphism T is called ergodic if every measurable set A that is invariant under $T(T^{-1}A = A)$ has either measure 0 or 1

Proposition (Equivalent definition)

An endomorphism T is ergodic if and only if every invariant function on $L^2(M)$ is a constant. That is 1 is a simple eigenvalue of U_T

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$\Rightarrow$ Ergodicity is a spectral property.

(To be precise on this point, a property of a measure preserving transformation is said to be spectral if whenever $U_T = WU_SW^{-1}$ for an invertible isometry W and a measure preserving transformation S, then S shares also this property.)

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If T is not ergodic, then it can be decomposed into ergodic components in the following sense.

- There exists a finest mod 0 partition ξ fixed under T (measurable + its elements C_i are invariant under T) and T is ergodic in its elements.
- T_{C_i} component of T: transformation induced by T in $C_i \in \xi$.
  (T_{C_i} is an endomorphism of C_i)

Periodicity

Definition

- *An endomorphism T is said to be periodic at a point $x \in M$ if $\exists\ p \in \mathbb{N}$ such that $T^p x = x$.*
- *T is said to be aperiodic if the set of points of periodicity has measure zero.*

- for $p \in \mathbb{N}$, $A_p = \{x \in M\ ;\ T^p x = x\}$: set of points at which T is periodic of period p.
- A_0: set of points at which T is aperiodic.

Remark

If the measure in M is continuous, then

Ergodicity $\Rightarrow$ aperiodicity.

The converse is false in general.

Ergodic version

Theorem (Eisner book)

Let $(\Omega, \mathcal{F}, \tau)$ be an ergodic measure preserving system, let $A \in \mathcal{F}$ with $\mu(A) > 0$ and $n \in \mathbb{N}$. Then there is a set $B \in \mathcal{F}$ such that $B, \tau B, ..., \tau^{n-1} B$ are pairwise disjoint and

$$\mu\left(\bigcup_{j=0}^{n-1} \tau^j B\right) \geq 1 - n\mu(A).$$

Ergodic case in anatomic spaces

Another version of Kakutani Rokhlin lemma was given in Petersen book requiring the conditions of ergodicity and the anatomicity of the system.

Theorem (Petersen book)

Let $\tau : \Omega \to \Omega$ be an ergodic measure preserving transformation on a nonatomic measure space $(\Omega, \mathcal{F}, \mu)$, n a positive integer and $\epsilon > 0$. Then there is a measurable set $B \in \mathcal{F}$ such that $B, \tau B, ..., \tau^{n-1}B$ are pairwise disjoint and cover Ω up to a set of measure less than ϵ.

Aperiodic case

Proposition

If T is an aperiodic endomorphism, then for any positive ϵ there exists a measurable set A of measure less than ϵ such that a finite number of the sets $T^{-k}A$ cover M.

The hypothesis of ergodicity in Kakutani-Rokhlin Lemma was weakened in Rokhlin paper as follows.

Proposition

If T is an aperiodic automorphism, then for any naturel number n and any positive ϵ, there exists a measurable set $A \subset M$ such that the sets $A, TA, ..., T^{-n}A$ are pairwise disjoint and the complement of their union has measure less than ϵ.

$\rightarrow$ The requirement of ergodicity is redundant and only aperiodicity is needed.

Mixing endomorphisms

Definition

We say that the endomorphism T is mixing on the sets $A_0, \dots, A_r$ if, for any sequence of complexes of non negative integers $(k_1^0, \dots, k_1^r), (k_2^0, \dots, k_2^r), \dots$ satisfying $\lim_{n\to\infty} \min_{0\leq i<j\leq r} |k_n^j - k_n^i| = \infty$ we have

$$\lim_{n\to\infty} \mu\Big(\bigcap_{i=0}^{r} T^{-k_n^i} A_i\Big) = \prod_{i=0}^{r} \mu(A_i).$$

Definition

We say that the endomorphism T is mixing on the bounded functions $f_0, \ldots, f_r$ if, for every such subsequence of complexes

$$\lim_{n\to\infty} < \prod_{i=0}^{r} U_T^{k_n^i} f_i, 1 >= \prod_{i=0}^{r} < f_i, 1 > .$$

- Mixing on the sets $A_0, \ldots, A_r \iff$ Mixing on the characteristic functions of these sets.
- Mixing of degree r = Mixing on any measurable sets $A_0, \ldots, A_r$ = Mixing on any bounded measurable functions $f_0, \ldots, f_r$.

- Mixing of degree 1 = Mixing.
  A transformation T of a probability measure space (X, μ) is said to be mixing if for any measurable sets $A, B \subset X$

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

- Mixing $\implies$ ergodicity.
- Mixing is a spectral property.

Thank you for your attention !