# Vector subspaces and operators with the Stone condition

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# Outline

# Frame Title

The objective of this presentation is to show that when we consider a uniformly complete $\Phi$ algebra $E$, and we consider a continuous function of $\mathcal{C}(\mathbb{R})$ then we can define the function $f$ on unital bounded sub-algebra of $E$. this result exists in Azouzi Youssef and Fethi Ben Amor paper "Vector subspaces and operators with the Stone condition".

# Riesz space

## Definition (Riesz space)

We say that $E$ is a Riesz space whenever $E$ is a real vector space and $E$ is a lattice (that is, whenever the supremum of each pair of members of $E$ exists and the infimum of each pair of members of $E$ exists), where, for each $f, g \in E$, the supremum of $f$ and $g$ is denoted $f \vee g$ and the infimum of $f$ and $g$ is denoted $f \wedge g$.

## Definition

A linear operator $T$ from a Riesz space $E$ into a Riesz space $F$ is called Riesz homomorphism if

$$T(f \vee g) = T(f) \vee T(g)$$

holds for all $f$ and $g$ in $E$.

## Definition (Monotone sequence)

Let $E$ be a Riesz space and let $(f_n)_{\in \mathbb{N}}$ be a sequence in $E$, then $f_n$ is increasing (resp., decreasing), denoted $f_n \uparrow$ (resp., $f_n \downarrow$), if $f_n \leq f_{n+1}$ (resp., $f_{n+1} \leq f_n$), for each $n \in$. Furthermore, $f_n \uparrow f$ (resp., $f_n \downarrow f$), if $f_n \uparrow$ and $f = \sup f_n n \in$ (resp., $f_n \downarrow$ and $f = \inf f_n n \in$).

## Definition (order convergence)

A sequence $(f_n)_{n \in \mathbb{N}}$ in $E$ is said to converge in order to $f$ if there exists a sequence $(P_n) \downarrow 0$ such that $|f_n - f| \leq P_n$ holds for all $n$.

## Definition (u-uniformly convergence)

A sequence $(f_n)_{n \in \mathbb{N}}$ in $E$ is said to converge $u$-uniformly to $f$ if there exists a sequence of numbers $(\epsilon_n) \downarrow 0$ such that $|f_n - f| \leq \epsilon_n u$ holds for all $n$.

### Definition (u-uniform Cauchy sequences)

A sequence $(f_n)_{n\in\mathbb{N}}$ in $E$ is called an u-uniform Cauchy sequence if for any $\epsilon > 0$ there exists an index $n_\epsilon$ such that $|f_n - f_m| \leq u\epsilon$ for all $n, m \geq n_\epsilon$.

### Definition (relatively uniformly convergence)

A sequence $(f_n)_{n\in\mathbb{N}}$ in $E$ is said to converge relatively uniformly to $f$ if there exists an element $u > 0$ in $E$ such that $(f_n)_{n\in\mathbb{N}}$ is $u$-uniformly convergent to $f$.

### Definition (uniformly complete Riesz spaces)

An Archimedean Riesz space is said to be uniformly complete, if for every $u > 0$ in $E$, every u-uniform Cauchy sequence has a limit.

# $f$-algebra

**Definition (Riesz algebra)**

A Riesz space E is called a Riesz algebra (lattice ordered algebra) if there exists in $E$ an associative multiplication with the usual algebra properties and such that $uv \geq 0$ for all $0 \leq u, v \in E$.

**Definition ($f$-algebra)**

A Riesz algebra $E$ is called $f$-algebra if it has the additional property that $u \wedge v = 0$ in $E$ implies $(uw) \wedge v = (wu) \wedge v = 0$ for all $0 \leq w \in E$.

**Remark**

An $f$-algebra with unit is $\Phi$-algebra.

**Definition (separating set)**

a set of functions $S$ from a set $D$ to a set $C$ is called a separating set for $D$ or said to separate the points of $D$ if for any two distinct elements $x$ and $y$ of $D$, there exists a function $f$ in $S$ so that $f(x) \neq f(y)$.

**Theorem (Stone–Weierstrass Theorem)**

*Suppose $X$ is a compact Hausdorff space and $A$ is a sub-algebra of $\mathcal{C}(X, \mathbb{R})$ which contains a non-zero constant function. Then $A$ is dense in $\mathcal{C}(X, \mathbb{R})$ if and only if it separates points.*

# Operating function

- Let $E$ be a $\Phi$-algebra in which the unit is denoted by $e$.
- $H_m(A)$ will denote the set of real-valued multiplicative lattice homomorphisms on $A$ where $A$ is an $f$-sub-algebra of $E$.

**Definition**

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function and let $u \in E$. We say that $f(u)$ exists in $E$ whenever there exists $b \in E$ and an $f$-sub-algebra $A$ of $E$ with $u, b \in A$ and $H_m(A)$ separating the points of A, such that $w(b) = f(w(u))$ for all $w \in H_m(A)$. There is at most one such $b$; it will be denoted by $f(u)$.

**Lemma**

Let $E$ be an Archimedean semiprime $f$-algebra and $A$ a countable subset of $E$. Then $H_m(A)$ separates the points of $\ll A \gg$ .

# operating function on a non-empty subset

Throughout this section we consider a uniformly complete $\Phi$-algebra $E$. For a none-empty subset $D$ of $E$, we will use the notation

- $\mathcal{A}(D) = \{f \in C(\mathbb{R}) : f(u)$ exists in $E$ for all $u \in D\}$

**Definition**

A function $f \in \mathcal{A}(D)$ is said to operate on $D$ if $f(u) \in D$ for all $u \in D$

**Theorem**

If $u$ is bounded, then $\mathcal{A}(u) = \mathcal{C}(\mathbb{R})$ and $f(u)$ is bounded for all $f \in \mathcal{C}(\mathbb{R})$.

### Lemma

*Let* $\mathcal{A}(D) = \{f \in C(\mathbb{R}) : f(u)$ *exists in* $E$ *for all* $u \in D\}$ *and let* $B$ *be a subalgebra of* $E$*. for every* $f, g \in \mathcal{A}(D)$ *and every* $a, b \in \mathbb{R}$*, we have:*

- $a \leq b \iff \forall \psi \in H_m(B), \psi(a) \leq \psi(b).$
- $f \leq g \implies f(u) \leq g(u).$

## Proof (Theorem).

Assume that $u$ is bounded.
$\mathcal{A}(u)$ is a ru-closed sub-algebra of $\mathcal{C}(\mathbb{R})$ :

✐ Let $f, g \in \mathcal{A}(u)$ and $B = \ll e, u, f(u), g(u) \gg$ the f-subalgebra of $E$ generated by $e, u, f(u)$ and $g(u)$.

$H_m(B)$ separates the points of $B$ and for all $w \in H_m(B)$ using the properties of multiplicative lattice homomorphisms we have:

✍

$$\begin{aligned}(f+g)(w(u)) &= f(w(u)) + g(w(u)) \\ &= w(f(u)) + w(g(u)) \\ &= w(f(u) + g(u))\end{aligned}$$

Then $(f+g)(u)$ exists and it is equal to $f(u) + g(u)$.

□

## Proof.

✍

$$
\begin{aligned}
(fg)(w(u)) &= f(w(u))g(w(u)) \\
&= w(f(u))w(g(u)) \\
&= w(f(u)g(u))
\end{aligned}
$$

Then $(fg)(u)$ exists and it is equal to $f(u)g(u)$.

✍

$$
\begin{aligned}
(f \wedge g)(w(u)) &= f(w(u)) \wedge g(w(u)) \\
&= w(f(u)) \wedge w(g(u)) \\
&= w(f(u) \wedge g(u))
\end{aligned}
$$

Then $(f \wedge g)(u)$ exists and it is equal to $f(u) \wedge g(u)$.

□

**Proof.**

- ✍ $\mathcal{A}(u)$ contains the constant functions. Let $B = \ll u \gg$ then $H_m(B)$ separates the points of $B$ and for all $w \in H_m(B)$ we have:

$$\mathbf{1}(w(u)) = 1 = w(e).$$

  so that $\mathbf{1}(u)$ exists and it is equal to $e$.

- ✍ Let $(f_n)$ be a sequence of elements of $\mathcal{A}(u)$ ru-convergent to $f$ in $E$. Since $u$ is bounded , $(f_n)$ is ru-Cauchy.

Observe also that the function $t \to t$ is in $\mathcal{A}(u)$ and then $\mathcal{A}(u)$ separates the points of . By the Stone-Weierstrass theorem $\mathcal{A}(u) = \mathcal{C}(\mathbb{R})$

□

## Proof.

Now let $f \in \mathcal{C}(\mathbb{R})$ and $\alpha \in \mathbb{R}_+$ such that $|u| \leq \alpha e$.
Since $f$ is continuous there exists $M \geq 0$ such that $|f(t)| \leq M$ for all $t \in [-\alpha, \alpha]$.
Now put $B = \ll e, u, f(u) \gg$ the f-subalgebra of $E$ generated by $e, u$ and $f(u)$ and observe that for all $\psi \in H_m(B)$ :

$$\begin{aligned} |\psi(u)| &= \psi(|u|) \\ &\leq \psi(\alpha e) = \alpha\psi(e) = \alpha \end{aligned}$$

It follows that

$$\begin{aligned} \psi(|f(u)|) &= |\psi(f(u))| \\ &= |f(\psi(u))| \leq M = \psi(Me) \end{aligned}$$

Since $H_m(B)$ separates the points of $B$, we obtain $|f(u)| \geq Me$ and $f(u)$ is bounded.

□

## Theorem

*For $u, v$ in $E$ :*
*If $f \in \mathcal{A}(u) \cap \mathcal{A}(v)$ is increasing, then $f \in \mathcal{A}(u \wedge v)$,*
*and $f(u \wedge v) = f(u) \wedge f(v)$.*

## Proof.

Let $B = \ll e, u, v, f(u), f(v) \gg$ be the subalgebra of $E$ generated by $e, u, v, f(u)$ and $f(v)$. For all $\psi \in H_m(B)$ :

$$\begin{aligned} \psi(f(u) \wedge f(v)) &= \psi(f(u)) \wedge \psi(f(v)) \\ &= f(\psi(u)) \wedge (f(\psi(v)) \\ &= f(\psi(u) \wedge \psi(v)) \\ &= f(\psi(u \wedge v)). \end{aligned}$$

Since $H_m(B)$ separates the points of $B$, we obtain the desired result. □

# References I

[1] Youssef Azouzi and Fethi Ben Amor. Vector subspaces and operators with the stone condition. *Positivity*, 14(4):585–593, 2010.

[2] GJHM Buskes, B De Pagter, and A van Rooij. Functional calculus on riesz spaces. *Indagationes Mathematicae*, 2(4):423–436, 1991.