Free Banach lattices and associated function spaces

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Plan of talk

(1) Introduction: Banach lattices; freeness.

(2) Definition and basic properties of $\mathrm{FBL}^{(p)}[E]$ (the free p-convex Banach lattice over E), and some associated function spaces.

(3) Embeddings of Banach lattices.

- $\mathrm{FBL}[E]$ versus related function spaces.
- If $E \subset F$, how does $\mathrm{FBL}^{(p)}[E]$ sit inside $\mathrm{FBL}^{(p)}[F]$?

(4) Properties of $\mathrm{FBL}^{(p)}[E]$ and associated lattices reflect properties of E: RNP vs. Fatou-like properties.

(5) Main tool: functions defined via operators.

Vector lattices: a brief introduction

Throughout, our field of scalars is $\mathbb{R}$.

A vector lattice is a vector space X with positive cone X_+ ($x \leqslant y$ if $y - x \in X_+$), which is also a lattice: $x, y \in X$ have their max (least upper bound) $x \vee y$ and min (greatest lower bound) $x \wedge y$.
Can visualize vector lattices as spaces of functions.

For $x \in X$, we define its positive and negative parts: $x_+ = x \vee 0$, $x_- = (-x)_+$. . Then $x = x_+ - x_-$. Define the modulus of x: $|x| = x_+ + x_- = x \vee (-x)$.

A linear map $T : X \to Z$ between vector lattices is a lattice homomorphism if $\forall x, y \in X,\ T(x \vee y) = Tx \vee Ty$.

A normed lattice is a vector lattice, with "order-consistent" norm: $\|x\| \leqslant \|y\|$ if $|x| \leqslant |y|$. A Banach lattice is a complete normed lattice.

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Operations on Banach lattices

Suppose X, Y are Banach lattices. $T \in B(X, Y)$ is a lattice homomorphism if it preserves lattice operations: suffices to verify that $(Tx) \vee (Ty) = T(x \vee y)$, for any $x, y \in X$.
T is a lattice isomorphism (isometry) if it is invertible (surjective isometry), and both T and T^{-1} are lattice homomorphisms.
T is a lattice embedding (isometric lattice embedding) if $T(X)$ is a sublattice of Y, and $T : X \to T(X)$ is a lattice isomorphism (resp. lattice isometry).

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p-convexity

Definition

Suppose $1 \leqslant p \leqslant \infty$. A Banach lattice X is *p-convex with constant C* if $\left\|\left(\sum_i |x_i|^p\right)^{1/p}\right\| \leqslant C\left(\sum_i \|x_i\|^p\right)^{1/p}$ holds for any $x_1, \ldots, x_N \in X$.

• Any Banach lattice is 1-convex with constant 1 (triangle inequality).

• Any $C(K)$-space is ∞-convex, with constant 1.
If X is ∞-convex (with constant 1), then it is lattice isomorphic (lattice isometric) to a sublattice of $C(K)$.

• $L_q(\mu)$ is p-convex with constant 1 for $p \leqslant q$.
If $p > q$, and $\dim L_q(\mu) = \infty$, then $L_q(\mu)$ is not p-convex.

• If X is q-convex, then it is also p-convex for $p \leqslant q$.

• If X is q-convex with constant C, then it can be renormed to be q-convex with constant 1.

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Free p-convex Banach lattice over a Banach space

Definition

Suppose E is a Banach space, and $1 \leqslant p \leqslant \infty$. A free p-convex Banach lattice on E ($\mathrm{FBL}^{(p)}[E]$) is the unique Banach lattice X so that:

- X is p-convex with constant 1.
- There exists an isometry $\phi = \phi_{E,p} : E \to X$ so that $\phi_{E,p}(E)$ generates X as a Banach lattice.
- If Z is a Banach lattice, p-convex with constant 1, then any $T \in B(E, Z)$ extends to a lattice homomorphism $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to Z$ so that $\|\widehat{T}\| = \|T\|$, and $\widehat{T} \circ \phi_{E,p} = T$.

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If Z is p-convex with constant C, we can construct an extension $\widehat{T}$ with $\|\widehat{T}\| \leqslant C\|T\|$ (renorming makes Z p-convex with constant 1).

Notation: $\mathrm{FBL}[E] := \mathrm{FBL}^{(1)}[E]$ (for $p = 1$, Z is an arbitrary lattice).

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Embedding of $\mathrm{FBL}^{(p)}[E]$ into $\mathrm{FBL}^{(p)}[F]$

Suppose $i : E \to F$ is an isometric embedding.

$$\begin{tikzcd} \mathrm{FBL}^{(p)}[E] \arrow[r, dashed, "\bar{i}"] & \mathrm{FBL}^{(p)}[F] \\ E \arrow[u, hook, "\phi_E"] \arrow[r, "i"] & F \arrow[u, hook, "\phi_F"'] \end{tikzcd}$$

[OTTT]: $\bar{i}$ is an injective lattice homomorphism, not necessarily isomorphic. $\bar{i}(\mathrm{FBL}^{(p)}[E])$ is a sublattice of $\mathrm{FBL}^{(p)}[F]$, usually not a closed one.

Definition

A sublattice $Y \subset Z$ is regular if for any $A \subset Y$, whenever $\vee^Y A$ exists, then $\vee^Z A$ also exists, and equals $\vee^Y A$.

$\bar{i}(\mathrm{FBL}^{(p)}[E])$ is a regular sublattice of $\mathrm{FBL}^{(p)}[F]$ if either (i) $p = \infty$ [Azouzi + Dhifaoui] or (ii) $i(E)$ is complemented in F [OTTT].

Theorem

Suppose $i : \ell_2 \to L_\infty(\mu)$ is an isometric embedding. For $p \in [1, \infty)$, $\bar{i}(\mathrm{FBL}^{(p)}[\ell_2])$ is not a regular sublattice of $\mathrm{FBL}^{(p)}[L_\infty(\mu)]$.

Embedding of $\mathrm{FBL}^{(p)}[E]$ into $\mathrm{FBL}^{(p)}[F]$

Suppose $i : E \to F$ is an isometric embedding.

$$\begin{tikzcd} \mathrm{FBL}^{(p)}[E] \arrow[r, dashed, "\bar{i}"] & \mathrm{FBL}^{(p)}[F] \\ E \arrow[u, hook, "\phi_E"] \arrow[r, "i"] & F \arrow[u, hook, "\phi_F"'] \end{tikzcd}$$

[OTTT]: $\bar{i}$ is an injective lattice homomorphism, not necessarily isomorphic. $\bar{i}(\mathrm{FBL}^{(p)}[E])$ is a sublattice of $\mathrm{FBL}^{(p)}[F]$, usually not a closed one.

Definition

A sublattice $Y \subset Z$ is regular if for any $A \subset Y$, whenever $\vee^Y A$ exists, then $\vee^Z A$ also exists, and equals $\vee^Y A$.

$\bar{i}(\mathrm{FBL}^{(p)}[E])$ is a regular sublattice of $\mathrm{FBL}^{(p)}[F]$ if either (i) $p = \infty$ [Azouzi + Dhifaoui] or (ii) $i(E)$ is complemented in F [OTTT].

Theorem

Suppose $i : \ell_2 \to L_\infty(\mu)$ is an isometric embedding. For $p \in [1, \infty)$, $\bar{i}(\mathrm{FBL}^{(p)}[\ell_2])$ is not a regular sublattice of $\mathrm{FBL}^{(p)}[L_\infty(\mu)]$.

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Some properties involving upper bound

Definition

A Banach lattice Z is called:

- λ-Fatou if, whenever $A \subset Z_+$ is an upward-directed set, and $b = \vee A$, then $\|b\| \leqslant \lambda \sup_{a \in A} \|a\|$.
- monotonically bounded (m.b.) if, whenever $A \subset Z_+$ is upward-directed and $\sup_{a \in A} \|a\| < \infty$, then A has an upper bound.
- λ-m.b. if, whenever $A \subset Z_+$ is upward-directed and $\sup_{a \in A} \|a\| < \infty$, then A has an upper bound b, with $\|b\| \leqslant \lambda \sup_{a \in A} \|a\|$.

Observation. (1) m.b. $\Rightarrow$ λ-m.b., for some λ. (2) λ-m.b. $\Rightarrow$ λ-Fatou.

Proposition

Suppose E is a Banach space. Then $\dim E < \infty$ iff $\mathrm{FBL}^{(\infty)}[E]$ is λ-Fatou for some (any) $\lambda \geqslant 1$. Then it is also λ-m.b. for any $\lambda \geqslant 1$.

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Properties of free lattices versus underlying spaces

Proposition (O.+Taylor+Tradacete+Troitsky)

For a Banach space E, TFAE:

1. *E contains a complemented copy of ℓ_1 (copies of ℓ_1^n, uniformly complementably).*
2. $\mathrm{FBL}[E]$ *contains ℓ_1 as a sublattice (ℓ_1^n, uniformly as sublattices).*
3. $\mathrm{FBL}[E]$ *contains ℓ_1 as a complemented subspace (ℓ_1^n as a uniformly complemented subspaces).*

A Banach space E has the Radon-Nikodym Property (RNP) if any $T \in B(L_1(0,1), E)$ is representable – that is, $\exists g \in L_\infty((0,1), E)$ s.t. $\forall f \in L_1(0,1)$, $Tf = \int fg$.

Fact. Separable dual spaces, reflexive spaces have the RNP. Spaces containing $L_1(0,1)$ or c_0 fail the RNP.

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Upper bound properties in free lattices

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If E is a dual Banach space with the RNP, $p \in [1, \infty)$, then $\mathrm{FBL}^{(p)}[E]$ is 1-m.b. (hence 1-Fatou).

Proposition

If a separable E contains c_0 or $L_1(0,1)$, $p \in [1, \infty)$, then $\mathrm{FBL}^{(p)}[E]$ is not λ-Fatou for any λ.

If E contains c_0 or L_1, then it fails RNP. The converse is false. The canonical predual to James Tree space JT (denoted by JT_*) contains neither c_0 nor L_1, but still fails the RNP.

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Functional representation of $\mathrm{FBL}^{(p)}[E]$

Denote by $\mathbf{H}[E]$ the space of positively homogeneous functions on E^* ($f(tx^*) = tf(x^*)\ \forall t \geqslant 0$), weak* continuous on bounded sets.

$\mathbf{H}_p[E]$ consists of those $f \in \mathbf{H}[E]$ for which $\exists C > 0$ s.t. $\left(\sum_{i=1}^N |f(x_i^*)|^p\right)^{1/p} \leqslant C$ when $\|(x_i^*)_{i=1}^N\|_{p,\mathrm{weak}} \leqslant 1$. $\|f\|_p := \inf C$.

Here $\|(x_i^*)_{i=1}^N\|_{p,\mathrm{weak}} = \sup_{x \in \mathbf{B}(E)} \left(\sum_i |\langle x_i^*, x\rangle|^p\right)^{1/p}$.

Let $\phi_{E,p} : E \to \mathbf{H}[E] : x \mapsto \delta_x$; $\delta_x(x^*) = \langle x^*, x\rangle$.

Theorem (O., Taylor, Tradacete, Troitsky)

$\phi_{E,p} : E \to \mathbf{H}_p[E]$ is an isometry. $\mathrm{FBL}^{(p)}[E]$ is the Banach lattice generated by $\phi_{E,p}(E)$ in $\mathbf{H}_p[E]$.

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Spaces and ideals associated with $\mathrm{FBL}^{(p)}[E]$

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If Y is a vector lattice, a subspace $Z \subset Y$ is an ideal if $z \in Z$ whenever $|z| \leqslant |w|$, $w \in Z$ (it is a sublattice).

Let $\mathbf{I}_p[E]$ be the ideal in $\mathbf{H}_p[E]$ generated by $\mathrm{FBL}^{(p)}[E]$. $f \in \mathbf{I}_p[E]$ iff $|f| \leqslant g \in \mathrm{FBL}^{(p)}[E]$. $\overline{\mathbf{I}_p}[E]$ (the closure of $\mathbf{I}_p[E]$) is also an ideal.

Question

$\mathrm{FBL}^{(p)}[E] \subseteq \mathbf{I}_p[E] \subseteq \overline{\mathbf{I}_p}[E] \subseteq \mathbf{H}_p[E]$. *Are these inclusions proper?*

Theorem (O.+Taylor+Tradacete+Troitsky)

If either $\dim E < \infty$ *or* $p = \infty$, $\mathrm{FBL}^{(p)}[E] = \mathbf{H}_p[E]$.

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$\mathrm{FBL}^{(p)}[E] \subseteq \mathbf{I}_p[E] \subseteq \overline{\mathbf{I}_p}[E] \subseteq \mathbf{H}_p[E]$. *Are these inclusions proper?*

Theorem (O.+Taylor+Tradacete+Troitsky)

If either $\dim E < \infty$ *or* $p = \infty$, $\mathrm{FBL}^{(p)}[E] = \mathbf{H}_p[E]$.

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Obtained earlier by N. Laustsen and P. Tradacete, assuming E has a separable infinite dimensional quotient.

Theorem

(1) *If E is a dual Banach space with the RNP, then* $\mathbf{I}_p[E] = \mathbf{H}_p[E]$.
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• Suppose T is compact. Then $\phi_T \in \mathbf{H}_p[X] \Leftrightarrow T^* \in \Pi_p(X^*, Z^*)$, and $\|\phi_T\| = \pi_p(T^*)$.

• Suppose $X_0 \subset X$ is dense. We say that T is dual p-persistent relative to X_0 if $\exists c > 0$ s.t. $\forall x_1, \ldots, x_n \in X_0$, $\pi_p(T^*|_{\cap_i x_i^\perp}) \geqslant c$.

• If T is dual p-persistent, then $\phi_T \notin \overline{\mathbf{I}_p}[X]$. Consequently, if T is compact and dual p-persistent, $T^* \in \Pi_p$, then $\phi_T \in \mathbf{H}_p[X] \backslash \overline{\mathbf{I}_p}[X]$.

Example. If $c_0 \subset X$, then $\overline{\mathbf{I}_p}[X] \subsetneq \mathbf{H}_p[X]$ for $1 \leqslant p < \infty$.

Let $q_k : \ell_1^{2^k} \to \ell_\infty^k$ be the quotient mapping, let $\kappa_k = \pi_p(I_{\ell_1^k}) \sim \sqrt{k}$.

Let $T : \ell_1 = (\oplus_k \ell_1^{2^k})_1 \to (\oplus_k \ell_\infty^k)_0 = c_0 \hookrightarrow X : (\xi_k) \mapsto \sum_k \kappa_k^{-1} q_k$.

Then T is compact and dual p-persistent, and $\pi_p(T^*) = 1$. ■

Proof techniques: functions on X^* from operators

For $T \in B(Z, X)$, consider $\phi_T : X^* \to \mathbb{R} : x^* \mapsto \|T^* x^*\|$. Then:

• T compact $\Rightarrow \phi_T$ is weak* continuous on bounded sets.
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Proof techniques: order properties

If $\dim F < \infty$, $T \in B(F, X)$, then $\phi_T \in \mathrm{FBL}^{(p)}[X]$; $\phi_T = \vee_{f \in \mathbf{B}(F)} \left|\delta_{Tf}\right|$. Moreover, $\forall C > 1 \, \exists \psi \in \mathrm{FVL}[X]$ s.t. $\psi \leqslant \phi_T \leqslant C\psi$.

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If X contains $L_1(0,1)$ or c_0, then $\exists$ non-compact $T \in B(Z, X)$, $T^* \in \Pi_p$.

For $L_1(0,1) \subset X$, can take $T = j \circ id$, where $L_1(0,1) \overset{j}{\hookrightarrow} X$, and $id : L_\infty(0,1) \to L_1(0,1)$.

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Positive results via operators

Theorem

Suppose X is a dual Banach space, $L_1(0,1) \not\subset X$. Then $\mathbf{H}_p[X]$ is 1-m.b.

Remark. Such an X has weak RNP $\Leftrightarrow \ell_1 \not\subset X_*$.

Proof. Suppose $(f_i)_{i \in I}$ is an increasing net in $\mathbf{B}(\mathbf{H}_p[X])_+$.
$\forall i \exists T_i : X^* \to L_p(\mu_i)$ s.t. $\pi_p(T_i) \leqslant 1$, $\phi_{T_i} \geqslant f_i$
(that is $\forall x^* \in X^*$, $\|T_i x^*\| \geqslant f_i(x^*)$).

Ultraproduct argument + Pietsch Domination: $\exists S : X^* \to L_p(\mu)$ s.t.
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Let $T = S\big|_{X_*}$, then $\iota_p(T) \leqslant 1$. $\forall x_* \in X_*, i \in I$, $\|Tx_*\| \geqslant f_i(x_*)$.

X has wRNP $\Rightarrow$ T is compact. $\Rightarrow \phi_{T^{**}} : X^* \to \mathbb{R} : x^* \mapsto \|T^{**}x^*\|$ is weak* continuous on bounded sets. Thus $\forall i$ $\phi_{T^{**}} \geqslant f_i$. Indeed, for $x^* \in \mathbf{B}(X^*)$ find $(z_\alpha) \in \mathbf{B}(X_*)$, $\mathrm{weak}^* - \lim_\alpha z_\alpha = x^*$.

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Questions welcome!