

Free Banach lattices and associated function spaces

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Plan of talk

- (1) Introduction: Banach lattices; freeness.
- (2) Definition and basic properties of $\text{FBL}^{(p)}[E]$ (the free p -convex Banach lattice over E), and some associated function spaces.
- (3) Embeddings of Banach lattices.
 - $\text{FBL}[E]$ versus related function spaces.
 - If $E \subset F$, how does $\text{FBL}^{(p)}[E]$ sit inside $\text{FBL}^{(p)}[F]$?
- (4) Properties of $\text{FBL}^{(p)}[E]$ and associated lattices reflect properties of E : RNP vs. Fatou-like properties.
- (5) Main tool: functions defined via operators.

Vector lattices: a brief introduction

Throughout, our field of scalars is $\mathbb{R}$.

A **vector lattice** is a vector space X with positive cone X_+ ($x \leq y$ if $y - x \in X_+$), which is also a lattice: $x, y \in X$ have their max (least upper bound) $x \vee y$ and min (greatest lower bound) $x \wedge y$.

Can visualize vector lattices as spaces of functions.

For $x \in X$, we define its **positive and negative parts**: $x_+ = x \vee 0$, $x_- = (-x)_+$. Then $x = x_+ - x_-$. Define the **modulus of x** : $|x| = x_+ + x_- = x \vee (-x)$.

A linear map $T : X \rightarrow Z$ between vector lattices is a **lattice homomorphism** if $\forall x, y \in X$, $T(x \vee y) = Tx \vee Ty$.

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Operations on Banach lattices

Suppose X, Y are Banach lattices. $T \in B(X, Y)$ is a **lattice homomorphism** if it preserves lattice operations: suffices to verify that $(Tx) \vee (Ty) = T(x \vee y)$, for any $x, y \in X$.

T is a **lattice isomorphism** (**isometry**) if it is invertible (surjective isometry), and both T and T^{-1} are lattice homomorphisms.

T is a **lattice embedding** (**isometric lattice embedding**) if $T(X)$ is a sublattice of Y , and $T : X \rightarrow T(X)$ is a lattice isomorphism (resp. lattice isometry).

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p -convexity

Definition

Suppose $1 \leq p \leq \infty$. A Banach lattice X is p -convex with constant C if $\|(\sum_i |x_i|^p)^{1/p}\| \leq C (\sum_i \|x_i\|^p)^{1/p}$ holds for any $x_1, \dots, x_N \in X$.

- Any Banach lattice is 1-convex with constant 1 (triangle inequality).

- Any $C(K)$ -space is ∞ -convex, with constant 1.

If X is ∞ -convex (with constant 1), then it is lattice isomorphic (lattice isometric) to a sublattice of $C(K)$.

- $L_q(\mu)$ is p -convex with constant 1 for $p \leq q$.

If $p > q$, and $\dim L_q(\mu) = \infty$, then $L_q(\mu)$ is not p -convex.

- If X is q -convex, then it is also p -convex for $p \leq q$.

- If X is q -convex with constant C , then it can be renormed to be q -convex with constant 1.

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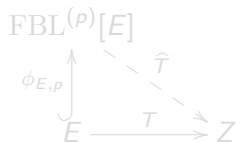
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Free p -convex Banach lattice over a Banach space

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Suppose E is a Banach space, and $1 \leq p \leq \infty$. A **free p -convex Banach lattice on E** ($\text{FBL}^{(p)}[E]$) is the unique Banach lattice X so that:

- X is p -convex with constant 1.
- There exists an isometry $\phi = \phi_{E,p} : E \rightarrow X$ so that $\phi_{E,p}(E)$ generates X as a Banach lattice.
- If Z is a Banach lattice, p -convex with constant 1, then any $T \in B(E, Z)$ extends to a lattice homomorphism $\hat{T} : \text{FBL}^{(p)}[E] \rightarrow Z$ so that $\|\hat{T}\| = \|T\|$, and $\hat{T} \circ \phi_{E,p} = T$.



If Z is p -convex with constant C , we can construct an extension $\hat{T}$ with $\|\hat{T}\| \leq C\|T\|$ (renorming makes Z p -convex with constant 1).

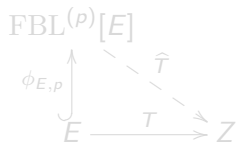
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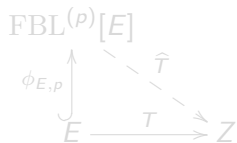
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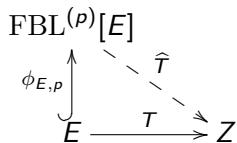
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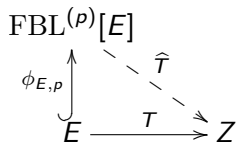
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Embedding of $\text{FBL}^{(p)}[E]$ into $\text{FBL}^{(p)}[F]$

Suppose $i : E \rightarrow F$ is an isometric embedding.

$$\text{FBL}^{(p)}[E] \xrightarrow{\bar{i}} \text{FBL}^{(p)}[F]$$

$$\begin{array}{ccc} \uparrow \phi_E & & \uparrow \phi_F \\ E & \xrightarrow{i} & F \end{array}$$

[OTTT]: $\bar{i}$ is an injective lattice homomorphism, not necessarily isomorphic. $\bar{i}(\text{FBL}^{(p)}[E])$ is a sublattice of $\text{FBL}^{(p)}[F]$, usually not a closed one.

Definition

A sublattice $Y \subset Z$ is **regular** if for any $A \subset Y$, whenever $\bigvee^Y A$ exists, then $\bigvee^Z A$ also exists, and equals $\bigvee^Y A$.

$\bar{i}(\text{FBL}^{(p)}[E])$ is a regular sublattice of $\text{FBL}^{(p)}[F]$ if either (i) $p = \infty$ [Azouzi + Dhifaoui] or (ii) $i(E)$ is complemented in F [OTTT].

Theorem

Suppose $i : \ell_2 \rightarrow L_\infty(\mu)$ is an isometric embedding. For $p \in [1, \infty)$, $\bar{i}(\text{FBL}^{(p)}[\ell_2])$ **is not** a regular sublattice of $\text{FBL}^{(p)}[L_\infty(\mu)]$.

Embedding of $\text{FBL}^{(p)}[E]$ into $\text{FBL}^{(p)}[F]$

Suppose $i : E \rightarrow F$ is an isometric embedding.

$$\text{FBL}^{(p)}[E] \xrightarrow{\bar{i}} \text{FBL}^{(p)}[F]$$

$$\begin{array}{ccc} \uparrow \phi_E & & \uparrow \phi_F \\ E & \xrightarrow{i} & F \end{array}$$

[OTTT]: $\bar{i}$ is an injective lattice homomorphism, not necessarily isomorphic. $\bar{i}(\text{FBL}^{(p)}[E])$ is a sublattice of $\text{FBL}^{(p)}[F]$, usually not a closed one.

Definition

A sublattice $Y \subset Z$ is **regular** if for any $A \subset Y$, whenever $\vee^Y A$ exists, then $\vee^Z A$ also exists, and equals $\vee^Y A$.

$\bar{i}(\text{FBL}^{(p)}[E])$ is a regular sublattice of $\text{FBL}^{(p)}[F]$ if either (i) $p = \infty$ [Azouzi + Dhifaoui] or (ii) $i(E)$ is complemented in F [OTTT].

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Some properties involving upper bound

Definition

A Banach lattice Z is called:

- **λ -Fatou** if, whenever $A \subset Z_+$ is an upward-directed set, and $b = \vee A$, then $\|b\| \leq \lambda \sup_{a \in A} \|a\|$.
- **monotonically bounded (m.b.)** if, whenever $A \subset Z_+$ is upward-directed and $\sup_{a \in A} \|a\| < \infty$, then A has an upper bound.
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Observation. (1) m.b. $\Rightarrow$ λ -m.b., for some λ . (2) λ -m.b. $\Rightarrow$ λ -Fatou.

Proposition

Suppose E is a Banach space. Then $\dim E < \infty$ iff $\text{FBL}^{(\infty)}[E]$ is λ -Fatou for some (any) $\lambda \geq 1$. Then it is also λ -m.b. for any $\lambda \geq 1$.

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Properties of free lattices versus underlying spaces

Proposition (O.+Taylor+Tradacete+Troitsky)

For a Banach space E , TFAE:

- 1 E contains a complemented copy of ℓ_1 (copies of ℓ_1^n , uniformly complementably).
- 2 $\text{FBL}[E]$ contains ℓ_1 as a sublattice (ℓ_1^n , uniformly as sublattices).
- 3 $\text{FBL}[E]$ contains ℓ_1 as a complemented subspace (ℓ_1^n as a uniformly complemented subspaces).

A Banach space E has the **Radon-Nikodym Property (RNP)** if any $T \in B(L_1(0,1), E)$ is **representable** – that is, $\exists g \in L_\infty((0,1), E)$ s.t. $\forall f \in L_1(0,1)$, $Tf = \int fg$.

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Upper bound properties in free lattices

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If E is a dual Banach space with the RNP, $p \in [1, \infty)$, then $\text{FBL}^{(p)}[E]$ is 1-m.b. (hence 1-Fatou).

Proposition

If a separable E contains c_0 or $L_1(0, 1)$, $p \in [1, \infty)$, then $\text{FBL}^{(p)}[E]$ is not λ -Fatou for any λ .

If E contains c_0 or L_1 , then it fails RNP. The converse is false. The canonical predual to James Tree space JT (denoted by JT_*) contains neither c_0 nor L_1 , but still fails the RNP.

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For $p \in [2, \infty)$, $\text{FBL}^{(p)}[JT_]$, $\text{FBL}^{(p)}[JT^*]$ are not λ -Fatou for any λ .*

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Functional representation of $\text{FBL}^{(p)}[E]$

Denote by $\mathbf{H}[E]$ the space of positively homogeneous functions on E^* ($f(tx^*) = tf(x^*) \forall t \geq 0$), weak* continuous on bounded sets.

$\mathbf{H}_p[E]$ consists of those $f \in \mathbf{H}[E]$ for which $\exists C > 0$ s.t.

$$\left(\sum_{i=1}^N |f(x_i^*)|^p\right)^{1/p} \leq C \text{ when } \|(x_i^*)_{i=1}^N\|_{p,\text{weak}} \leq 1. \quad \|f\|_p := \inf C.$$

Here $\|(x_i^*)_{i=1}^N\|_{p,\text{weak}} = \sup_{x \in B(E)} \left(\sum_i |\langle x_i^*, x \rangle|^p\right)^{1/p}$.

Let $\phi_{E,p} : E \rightarrow \mathbf{H}[E] : x \mapsto \delta_x; \delta_x(x^*) = \langle x^*, x \rangle$.

Theorem (O., Taylor, Tradacete, Troitsky)

$\phi_{E,p} : E \rightarrow \mathbf{H}_p[E]$ is an isometry. $\text{FBL}^{(p)}[E]$ is the Banach lattice generated by $\phi_{E,p}(E)$ in $\mathbf{H}_p[E]$.

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Spaces and ideals associated with $\text{FBL}^{(p)}[E]$

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If Y is a vector lattice, a subspace $Z \subset Y$ is an **ideal** if $z \in Z$ whenever $|z| \leq |w|, w \in Z$ (it is a sublattice).

Let $\mathbf{I}_p[E]$ be the ideal in $\mathbf{H}_p[E]$ generated by $\text{FBL}^{(p)}[E]$. $f \in \mathbf{I}_p[E]$ iff $|f| \leq g \in \text{FBL}^{(p)}[E]$. $\overline{\mathbf{I}_p[E]}$ (the closure of $\mathbf{I}_p[E]$) is also an ideal.

Question

$\text{FBL}^{(p)}[E] \subseteq \mathbf{I}_p[E] \subseteq \overline{\mathbf{I}_p[E]} \subseteq \mathbf{H}_p[E]$. Are these inclusions proper?

Theorem (O.+Taylor+Tradacete+Troitsky)

If either $\dim E < \infty$ or $p = \infty$, $\text{FBL}^{(p)}[E] = \mathbf{H}_p[E]$.

Spaces and ideals associated with $\text{FBL}^{(p)}[E]$

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$(\sum_{i=1}^N |f(x_i^*)|^p)^{1/p} \leq C$ when $\|(x_i^*)_{i=1}^N\|_{p,\text{weak}} \leq 1$. $\|f\|_p := \inf C$.

$\phi_{E,p} : E \rightarrow \mathbf{H}_p[E] : x \mapsto \delta_x$ is an isometry. $\text{FBL}^{(p)}[E]$ is the Banach lattice generated by $\phi_{E,p}(E)$ in $\mathbf{H}_p[E]$.

If Y is a vector lattice, a subspace $Z \subset Y$ is an **ideal** if $z \in Z$ whenever $|z| \leq |w|, w \in Z$ (it is a sublattice).

Let $\mathbf{I}_p[E]$ be the ideal in $\mathbf{H}_p[E]$ generated by $\text{FBL}^{(p)}[E]$. $f \in \mathbf{I}_p[E]$ iff $|f| \leq g \in \text{FBL}^{(p)}[E]$. $\overline{\mathbf{I}_p[E]}$ (the closure of $\mathbf{I}_p[E]$) is also an ideal.

Question

$\text{FBL}^{(p)}[E] \subseteq \mathbf{I}_p[E] \subseteq \overline{\mathbf{I}_p[E]} \subseteq \mathbf{H}_p[E]$. Are these inclusions proper?

Theorem (O.+Taylor+Tradacete+Troitsky)

If either $\dim E < \infty$ or $p = \infty$, $\text{FBL}^{(p)}[E] = \mathbf{H}_p[E]$.

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If $\dim E = \infty$, $p \in [1, \infty)$, then $\text{FBL}^{(p)}[E] \subsetneq \mathbf{I}_p[E]$.

Obtained earlier by N. Laustsen and P. Tradacete, assuming E has a separable infinite dimensional quotient.

Theorem

- (1) If E is a dual Banach space with the RNP, then $\mathbf{I}_p[E] = \mathbf{H}_p[E]$.*
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- Suppose T is compact. Then $\phi_T \in \mathbf{H}_p[X] \Leftrightarrow T^* \in \Pi_p(X^*, Z^*)$, and $\|\phi_T\| = \pi_p(T^*)$.
- Suppose $X_0 \subset X$ is dense. We say that T is **dual p -persistent** relative to X_0 if $\exists c > 0$ s.t. $\forall x_1, \dots, x_n \in X_0, \pi_p(T^*|_{\cap_i x_i^\perp}) \geq c$.
- If T is dual p -persistent, then $\phi_T \notin \overline{\mathbf{I}}_p[X]$. Consequently, if T is compact and dual p -persistent, $T^* \in \Pi_p$, then $\phi_T \in \mathbf{H}_p[X] \setminus \overline{\mathbf{I}}_p[X]$.

Example. If $c_0 \subset X$, then $\overline{\mathbf{I}}_p[X] \subsetneq \mathbf{H}_p[X]$ for $1 \leq p < \infty$.

Let $q_k : \ell_1^{2^k} \rightarrow \ell_\infty^k$ be the quotient mapping, let $\kappa_k = \pi_p(I_{\ell_1^{2^k}}) \sim \sqrt{k}$.

Let $T : \ell_1 = (\oplus_k \ell_1^{2^k})_1 \rightarrow (\oplus_k \ell_\infty^k)_0 = c_0 \hookrightarrow X : (\xi_k) \mapsto \sum_k \kappa_k^{-1} q_k$.

Then T is compact and dual p -persistent, and $\pi_p(T^*) = 1$. ■

Proof techniques: functions on X^* from operators

For $T \in B(Z, X)$, consider $\phi_T : X^* \rightarrow \mathbb{R} : x^* \mapsto \|T^*x^*\|$. Then:

- T compact $\Rightarrow \phi_T$ is weak* continuous on bounded sets.
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If $\dim F < \infty$, $T \in B(F, X)$, then $\phi_T \in \text{FBL}^{(p)}[X]$; $\phi_T = \bigvee_{f \in \mathbf{B}(F)} |\delta_{Tf}|$.
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If $T \in B(Z, X)$ is not compact, $T \in \Pi_p^*$, then

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For $L_1(0, 1) \subset X$, can take $T = j \circ id$, where $L_1(0, 1) \xrightarrow{j} X$, and
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Positive results via operators

Theorem

Suppose X is a dual Banach space, $L_1(0,1) \not\subset X$. Then $\mathbf{H}_p[X]$ is 1-m.b.

Remark. Such an X has weak RNP $\Leftrightarrow \ell_1 \not\subset X_*$.

Proof. Suppose $(f_i)_{i \in I}$ is an increasing net in $\mathbf{B}(\mathbf{H}_p[X])_+$.

$\forall i \exists T_i : X^* \rightarrow L_p(\mu_i)$ s.t. $\pi_p(T_i) \leq 1$, $\phi_{T_i} \geq f_i$

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Ultraproduct argument + Pietsch Domination: $\exists S : X^* \rightarrow L_p(\mu)$ s.t.

$\iota_p(S) \leq 1$, $\|Sx^*\| \geq f_i(x^*) \forall i \in I, x^* \in X^*$.

Let $T = S|_{X_*}$, then $\iota_p(T) \leq 1$. $\forall x_* \in X_*, i \in I$, $\|Tx_*\| \geq f_i(x_*)$.

X has wRNP $\Rightarrow T$ is compact. $\Rightarrow \phi_{T^{**}} : X^* \rightarrow \mathbb{R} : x^* \mapsto \|T^{**}x^*\|$ is weak* continuous on bounded sets. Thus $\forall i$ $\phi_{T^{**}} \geq f_i$. Indeed, for $x^* \in \mathbf{B}(X^*)$ find $(z_\alpha) \in \mathbf{B}(X_*)$, $\text{weak}^* - \lim_\alpha z_\alpha = x^*$.

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Questions welcome!

