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A Nakano-type theorem in pervasive pre-Riesz spaces

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The problem

Setting.

Let X, Y be ordered vector spaces (ovs) with X directed and Y Archimedean. Then

$$\mathrm{L}^{\mathrm{r}}(X;Y), \mathrm{L}^{\mathrm{r}}_{\mathrm{n}}(X;Y)$$

are pre-Riesz spaces. Even if X, Y are Archimedean vector lattices, $\mathrm{L}^{\mathrm{r}}(X;Y)$ and $\mathrm{L}^{\mathrm{r}}_{\mathrm{n}}(X;Y)$ need not be vector lattices, and it is not clear if $\mathrm{L}^{\mathrm{r}}_{\mathrm{n}}(X;Y)$ is a band in $\mathrm{L}^{\mathrm{r}}(X;Y)$!!!

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Which operators $T \in \mathrm{L}^{\mathrm{r}}(X;Y)$ have a modulus? If the modulus exists, is it then given by the Riesz–Kantorovich formula?

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Existence of the modulus in pre-Riesz spaces

Theorem (Kalauch, van Gaans, S., 2021).

Let X be a pre-Riesz space. For $x \in X$, the following are equivalent:

(i) $|x|$ exists in X.

(ii) $x \in X_+ \cup (-X_+)$ or there exist $a, b \in X_+ \setminus \{0\}$ with $a \perp b$ such that $x = a - b$.

Alternative problem.

Characterize the operators $S, T \in \mathrm{L}_+(X; Y)$ with $S \perp T$.

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Solution 1: Embedding in larger operator spaces

Definition/Proposition (Wickstead, 2024).

Let V be an Archimedean vector lattice and $F \subseteq V$ an order dense sublattice. F is *sequentially order closed in* V if the following are satisfied:

(1) Any increasing sequence in F with a supremum in F has the same supremum in V.

(2) If (x_n) is an increasing sequence in F which is bounded above, then (x_n) has a supremum in V.

V is the sequential order closure of F if (2) does not hold for any smaller sublattice of V.

F always has a sequential order closure, which is essentially unique. We denote it by F^σ.

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Theorem (Wickstead, 2024).

Let F be an Archimedean vector lattice. Then the Riesz completion of $\mathrm{L}^{\mathrm{r}}(\ell_0^\infty; F)$ can be identified with the space of operators

$$\Sigma := \{T \in \mathrm{L}^{\mathrm{r}}(\ell_0^\infty; F^\sigma) \mid Te_n \in F \ \forall n \in \mathbb{N}\},$$

that is, there exists a bipositive linear map

$$\Phi\colon \mathrm{L}^{\mathrm{r}}(\ell_0^\infty; F) \to \Sigma$$

such that $\Phi(\mathrm{L}^{\mathrm{r}}(\ell_0^\infty; F))$ is order dense in Σ and generates Σ as a vector lattice.

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Theorem (Starkey, Xanthos, 2025)

Let E, F be Archimedean vector lattices. If $\mathrm{L}^{\mathrm{r}}(E;F)$ is pervasive in $\mathrm{L}^{\mathrm{r}}(E;F^{\delta})$, then $\mathrm{L}^{\mathrm{r}}(E;F)^{\rho}$ can be identified with the Riesz subspace of $\mathrm{L}^{\mathrm{r}}(E;F^{\delta})$ generated by $\mathrm{L}^{\mathrm{r}}(E;\iota(F))$.
Moreover, if for $T \in \mathrm{L}^{\mathrm{r}}(E;F)$ the modulus exists, then it is given by the Riesz–Kantorovich formula. And, $\mathrm{L}^{\mathrm{r}}_{\mathrm{n}}(E;F)$ is a band of $\mathrm{L}^{\mathrm{r}}(E;F)$.

Proposition (Starkey, Xanthos, 2025).

If one of the following is satisfied, then $\mathrm{L}^{\mathrm{r}}(E;F)$ is pervasive in $\mathrm{L}^{\mathrm{r}}(E;F^{\delta})$:

(1) The space of rank-one operators $\mathcal{RO}(E;F^{\delta})$ is pervasive in $\mathrm{L}^{\mathrm{r}}(E;F^{\delta})$.

(2) F is atomic.

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Solution 2: Embedding in spaces of continuous functions

Theorem (Glück, Kalauch, van Gaans, 2025).

Let X, Y be non-zero ordered normed spaces such that:

(1) The cone X_+ is total, there exists an equivalent norm on X that is additive on X_+, and every extremal vector of X_+ is also extremal in the bidual cone X''_+. Moreover, the convex hull of the extremal vectors in X_+ is dense in X_+.

(2) The cone Y_+ has non-empty interior.

Then the closure $\mathcal{C}(X;Y)$ of the subspace of all finite-rank operators in $\mathcal{L}(X;Y)$ can be order densely and bipositively embedded into some $C(\Omega)$-space.

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The Nakano theorem

Definition.

Let E, F be vector lattices with F Dedekind complete and $T \in \mathrm{L}^r(E;F)$. The null ideal N_T of T is defined by

$$N_T := \{x \in E \mid |T|(|x|) = 0\}.$$

The carrier C_T of T is defined as $C_T := N_T^{\mathrm{d}}$.

The Nakano theorem

Theorem (Nakano, 1950).

Let E be an Archimedean vector lattice. For $\varphi, \psi \in E_n^\sim$, the following are equivalent:

(i) $\varphi \perp \psi$.

(ii) $C_\varphi \subseteq N_\psi$.

(iii) $C_\psi \subseteq N_\varphi$.

(iv) $C_\varphi \perp C_\psi$.

The null ideal of a positive operator in ovs

Definition.

Let X, Y be ovs and $T \in \mathrm{L}_+(X; Y)$. We define $s_{T,x} := \inf\{T(v);\ \pm x \leq v\}$ and call

$$N_T := \{x \in X;\ s_{T,x} = 0\}$$

the *null ideal of* T. The *carrier* C_T *of* T is defined as $C_T := N_T^{\mathrm{d}}$.

Proposition (S., 2026).

Let X, Y be ovs and $T \in \mathrm{L}_+(X; Y)$.

(a) N_T is an ideal and we have $\ker T \cap X_+ \subseteq N_T \subseteq \ker T$.

(b) N_T is directed if and only if $s_{T,x}$ is attained for every $x \in N_T$.

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The null ideal of a positive operator in ovs

Example that N_T can be non-directed (Glück).

Take $X = \mathrm{C}^1([-1,1])$ with pointwise order, and the functional $\delta_0 \colon X \to \mathbb{R}$, $\delta_0(x) = x(0)$. Then we have $\ker \delta_0 \cap X_+ = N_{\delta_0} \cap X_+ = \{x \in X_+;\ x(0) = x'(0) = 0\}$. Thus, $\mathrm{id}_{[-1,1]} \notin (N_{\delta_0} \cap X_+) - (N_{\delta_0} \cap X_+)$, but $\mathrm{id}_{[-1,1]} \in N_{\delta_0}$.

Proposition (S., 2026).

Let X, Y be ovs and $T \in \mathrm{L}_+(X;Y)$.

(a) If Y is Archimedean, then N_T is relatively uniformly closed.

(b) If T is order continuous, then N_T is order closed.

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(a) If Y is Archimedean, then N_T is relatively uniformly closed.

(b) If T is order continuous, then N_T is order closed.

The null ideal of a positive operator in ovs

Example that N_T need not be a band if T is order continuous.

Let $X = \mathbb{R}^3$ be endowed with the ice-cream cone. Then any positive functional is order continuous, in particular, the functional

$$\varphi\colon X \to \mathbb{R}, (x_1, x_2, x_3)^\top \mapsto x_3 - x_1.$$

In this case, $N_\varphi = \ker\varphi = \{(x_1, x_2, x_3)^\top \in X \mid x_1 = x_3\}$ is not a band (as there are non-trivial bands in X).

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(i) $\varphi \perp \psi$.

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An extension result for order continuous operators

Theorem (Bilokopytov, 2025).

Let X be an Archimedean pervasive pre-Riesz space and F a Dedekind complete vector lattice. Let $T \in \mathrm{L}_{\mathrm{n}}^{\mathrm{r}}(X;F) \cap \mathrm{L}_{+}(X;F)$. Then, for every $y \in X_{+}^{\rho}$,

$$\widetilde{T}(y) := \sup\{T(x) \mid x \in X,\ 0 \leq \iota_X(x) \leq y\}$$

defines an operator $\widetilde{T} \in \mathrm{L}_{\mathrm{n}}^{\mathrm{r}}(X^{\rho};F) \cap \mathrm{L}_{+}(X^{\rho};F)$ that extends T.

$$\begin{tikzcd} X^{\rho} \arrow[dr, "\widetilde{T}"] & \\ X \arrow[u, hook, "\iota_X"] \arrow[r, "T"'] & F \end{tikzcd}$$

Properties of the extension

Let X be an Archimedean pervasive pre-Riesz space and F a Dedekind complete vector lattice.

Proposition (S., 2026).

Let $T \in \mathrm{L}^{\mathrm{r}}_{\mathrm{n}}(X;F) \cap \mathrm{L}_{+}(X;F)$. Then:

(a) N_T is a band and $\iota_X^{-1}[N_{\widetilde{T}}] = N_T$.

(b) C_T is a band and $\iota_X^{-1}[C_{\widetilde{T}}] = C_T$.

Proposition (S., 2026).

Let $S, T \in \mathrm{L}^{\mathrm{r}}_{\mathrm{n}}(X;F) \cap \mathrm{L}_{+}(X;F)$. The following are equivalent:

(i) $S \perp T$ in $\mathrm{L}^{\mathrm{r}}_{\mathrm{n}}(X;F)$.

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A Nakano-type theorem

Theorem (S, 2026).

Let X be an Archimedean pervasive pre-Riesz space and $\varphi, \psi \in X_{\mathrm{n}}^* \cap X_+^*$. Then the following statements are equivalent:

(i) $\varphi \perp \psi$ in $\mathrm{L}_{\mathrm{n}}^{\mathrm{r}}(X; F)$.

(ii) $C_\varphi \subseteq N_\psi$.

(iii) $C_\varphi \subseteq N_\psi$.

(iv) $C_\varphi \perp C_\psi$.

Open question.

Do we also have that $\varphi \perp \psi$ in $\mathrm{L}^{\mathrm{r}}(X; F)$?

Thank you very much!
Merci!
Dankeschön!

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