Pre-Riesz spaces -
what we know, what we don't know . . .

Anke Kalauch

TU Dresden, Germany

CIRM 2026

A pre-Riesz space X is an ordered vector space such that

- there are a vector lattice Y and a bipositive linear map $i\colon X \to Y$,
- $i[X]$ is order dense in Y.

Y is called a Riesz cover of X.

Examples:

- Riesz spaces (classical sequence spaces and function spaces that are naturally equipped with the entry-wise partial order; Banach lattices, ...)
- ordered Banach spaces (with generating and closed cone)
- spaces of differentiable functions
- self-adjoint part of non-commutative C^*-algebras
- spin factors, etc.

History

- 1993: van Haandel, Buskes, van Rooij
- 2006: Disjointness in pre-Riesz spaces, vG., K.
- Last 20 years: van Gaans, Lemmens, Roelands, Glueck, Kusraeva, K.
- Malinowski, Zhang, Hauser, Stennder, Boisen, Raickwade

1. Every Archimedean directed ordered vector space has a Dedekind completion, and the embedding is order dense. But, in examples, the Dedekind completion is "too big" and not accessible.
2. In the theory of Riesz spaces or Banach lattices, one has situations (subspaces, spaces of operators, . . .) where one looses the lattice structure.
3. Task: Develop a theory of substructures and structure preserving operators in pre-Riesz spaces!

Outline

1. Structures in pre-Riesz spaces: Ideals, disjointness, bands, . . . in the directed and non-directed version
2. Operators on pre-Riesz spaces: Orthomorphisms, Riesz* homomorphisms, disjointness pres. operators, etc.
3. The importance of the Meyer theorem (dp. C_0-semigroups) . . .

More open problems:

- the localization principle,
- factorization with respect to ideals,
- vector lattice covers for pre-Riesz spaces of operators,
- the modulus of operators and the Riesz -Kantorovich formula, etc.

Ordered vector spaces

Let X be a (real) vector space. A partial order $\leq$ on X is called a vector space order if

(a) $x, y, z \in X$ and $x \leq y$ imply $x + z \leq y + z$,

(b) $x \in X$, $0 \leq x$ and $\lambda \in [0, \infty)$ imply $0 \leq \lambda x$.

The set $X_+ := \{x \in X; 0 \leq x\}$ is then a cone in X, i.e., $x, y \in X_+$, $\lambda \in [0, \infty)$ imply $\lambda x + y \in X_+$, and $X_+ \cap (-X_+) = \{0\}$.

X is then called an ordered vector space (ovs).

Remark: $i[X]$ being order dense in Y, i.e., for every $y \in Y$ one has

$$y = \inf\{i(x);\ x \in X,\ i(x) \geq y\},$$

is quite a weak property . . .

An ovs X is called Archimedean if, for every $x, y \in X$ such that $nx \leq y$ for all $n \in \mathbb{N}$, one has that $x \leq 0$.

X is directed (i.e., any two elements have a common upper bound) if and only $X = X_+ - X_+$.

Let $(X, X_+, \|\cdot\|)$ be an ordered normed space.

- X_+ is closed $\quad \Rightarrow \quad$ X is Archimedean
- X finite dimensional: "$\Leftrightarrow$"

- $\operatorname{int}(X_+) \neq \varnothing \quad \Rightarrow \quad$ X is directed
- X finite dimensional: "$\Leftrightarrow$"

Proposition (van Haandel, 1993)

- *Every Archimedean directed ovs is a pre-Riesz space.*
- *Every pre-Riesz space is directed.*

Substructures

1. disjointness: $x \perp y$, disjoint complement of $M \subseteq X$: M^{d}
2. $B \subseteq X$ band, i.e., $B = B^{\mathrm{dd}}$ (as in Archimedean Riesz spaces)
3. ideals, solvex ideals, ...

which then leads to structure preserving linear operators $T\colon X \to X$:

- disjointness preserving operators, i.e., $x \perp y$ implies $Tx \perp Ty$,
- band preserving operators, i.e., for every band B in X one has $T[B] \subseteq B$,
- orthomorphisms, i.e., order bounded band preserving operators, ...

Disjointness in pre-Riesz spaces

For $x, y \in X_+$, define $x \perp y$ whenever $x \wedge y = 0$.

X Riesz space: $\quad x \perp y \iff |x + y| = |x - y|$.

General trick: Replace suprema by sets of upper bounds!

For $M \subseteq X$, denote by M^{u} the set of upper bounds of M, and by M^{ℓ} the set of lower bounds of M.

X pre-Riesz space:

$$x \perp y \quad :\iff \quad \{x + y, -(x + y)\}^{\mathrm{u}} = \{x - y, -(x - y)\}^{\mathrm{u}}.$$

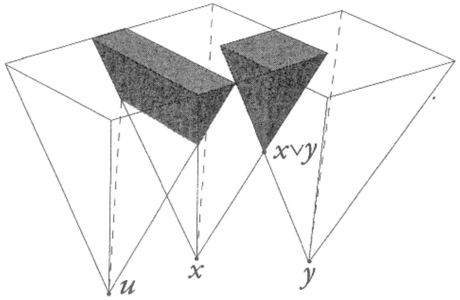
[vG., K., 2006]:

Proposition

$x \perp y$ in $X \iff i(x) \perp i(y)$ in Y.

Four-ray cone in $\mathbb{R}^3$

The supremum of x and y exists, the supremum of x and u does not exist.



How many non-trivial bands are there in this pre-Riesz space?

"Ideals" in pre-Riesz spaces

Archimedean vector lattice

$$
\begin{tikzcd}
\boxed{\begin{array}{l}\text{band}\\ = \text{o-closed ideal}\end{array}} \arrow[d, Rightarrow, shift right=2] \\
\boxed{\begin{array}{l}\text{ideal}\\ = \text{solvex ideal}\end{array}} \arrow[u, Rightarrow, shift right=2, "/" marking]
\end{tikzcd}
$$

pre-Riesz space

$$
\begin{tikzcd}
 & \boxed{\text{band}} \arrow[dl, Rightarrow, shift right] \arrow[dr, Rightarrow, shift left] & \\
\boxed{\text{o-closed ideal}} \arrow[ur, Rightarrow, shift right, "/" marking] \arrow[rr, Rightarrow, shift right, "/" marking] \arrow[dr, Rightarrow, shift left] & & \boxed{\text{solvex ideal}} \arrow[ul, Rightarrow, shift left, "/" marking] \arrow[ll, Rightarrow, shift right, "/" marking] \arrow[dl, Rightarrow, shift right] \\
 & \boxed{\text{ideal}} \arrow[ul, Rightarrow, shift left, "/" marking] \arrow[ur, Rightarrow, shift right, "/" marking] &
\end{tikzcd}
$$

Let X be an Archimedean ovs with order unit u and consider the order unit norm

$$\|\cdot\|_u : X \to [0, \infty), \quad x \mapsto \|x\|_u := \inf\{\lambda \in (0, \infty);\ -\lambda u \leq x \leq \lambda u\}.$$

The set

$$\Sigma := \{\varphi \in X'_+;\ \varphi(u) = 1\}$$

is a weakly-$*$ compact base of X'_+, hence convex.

Denote by $\Lambda := \mathrm{ext}(\Sigma)$ the set of all extreme points of Σ, and by $\overline{\Lambda}$ the weak-$*$ closure of Λ in Σ.

Λ is a total set. $\overline{\Lambda}$ is a compact Hausdorff space.

The bipositive linear map

$$i : X \to C(\overline{\Lambda}), \quad x \mapsto (\varphi \mapsto \varphi(x))$$

gives a Riesz cover of X [K., Lemmens, vG., 2014].

Counting bands in an order unit space comes down to counting bisaturated pairs (M_1, M_2), i.e., $M_1, M_2 \subseteq \overline{\Lambda}$ such that

$$\overline{\operatorname{aff}(M_1)} = \overline{\operatorname{aff}(\overline{\Lambda} \setminus M_2)} \quad \text{and} \quad \overline{\operatorname{aff}(M_2)} = \overline{\operatorname{aff}(\overline{\Lambda} \setminus M_1)}.$$

What we know [K., Lemmens, vG., 2015]:

Theorem

In an n-dimensional order unit space with $n \geq 2$, there are not more than $\frac{1}{4}2^{2^n}$ bands.

What we don't know:

- An optimal upper bound for dimension n?
- An (optimal) upper bound for directed bands in dim. n?

More generally, the problem can be formulated in the language of matroids, but AI does not know the answer (yet) . . .

Orthomorphisms on pre-Riesz spaces

…are not investigated yet. Many open problems!

For every Archimedean Riesz space X, $\mathrm{Orth}(X)$ under composition is an Archimedean f-algebra, having the identity operator I as unit.

What is the structure of $\mathrm{Orth}(X)$, if X is a pre-Riesz space?

Notice the following definition of [Stennder, 2025]:

An ordered algebra X with associative multiplication is called an f*-algebra if the multiplication is a bi-orthomorphism.

The zoo of "Riesz-homs." on pre-Riesz spaces

Let $T\colon X \to Y$ be a linear map.

If X and Y are **Riesz spaces**, then the following are equivalent:

$$
\begin{array}{llrcl}
(a) & \forall x_1, x_2 \in X: & T(x_1 \vee x_2) & = & T(x_1) \vee T(x_2) \\
\color{red}{(b)} & \forall x_1, x_2 \in X: & T\left[\{x_1, x_2\}^{\mathrm{u}\ell}\right]^{\mathrm{u}\ell} & = & \{T(x_1), T(x_2)\}^{\mathrm{u}\ell} \\
(b') & \forall x_1, x_2 \in X: & T\left[\{x_1, x_2\}^{\mathrm{u}\ell}\right] & \subseteq & \{T(x_1), T(x_2)\}^{\mathrm{u}\ell} \\
\color{red}{(c)} & \forall x_1, \ldots, x_n \in X: & T\left[\{x_1, \ldots, x_n\}^{\mathrm{u}\ell}\right]^{\mathrm{u}\ell} & = & \{T(x_1), \ldots, T(x_n)\}^{\mathrm{u}\ell} \\
(c') & \forall x_1, \ldots, x_n \in X: & T\left[\{x_1, \ldots, x_n\}^{\mathrm{u}\ell}\right] & \subseteq & \{T(x_1), \ldots, T(x_n)\}^{\mathrm{u}\ell} \\
\color{red}{(d)} & \forall x_1, x_2 \in X: & T\left[\{x_1, x_2\}^{\mathrm{u}}\right]^{\ell} & = & \{T(x_1), T(x_2)\}^{\mathrm{u}\ell} \\
(d') & \forall x_1, \ldots, x_n \in X: & T\left[\{x_1, \ldots, x_n\}^{\mathrm{u}}\right]^{\ell} & = & \{T(x_1), \ldots, T(x_2)\}^{\mathrm{u}\ell}
\end{array}
$$

If X, Y are **pre-Riesz spaces**, then

$\color{red}{(d)} \Leftrightarrow (d') \Rightarrow \color{red}{(c)} \Leftrightarrow (c') \Rightarrow \color{red}{(b)} \Leftrightarrow (b')$

(d) Riesz homomorphism [Buskes, van Rooij, 1993]

(c) Riesz* homomorphism [van Haandel, 1993]

(b) mild Riesz* homomorphisms
[Boisen, Hölker, K., Stennder, van Gaans, 2024]

The composition of (mild) Riesz* homomorphisms is a (mild) Riesz* homomorphism.

complete Riesz homomorphism

$\Downarrow$

Riesz homomorphism

$\Downarrow$

Riesz* homomorphism

$\Downarrow$

mild Riesz* homomorphism

$\Downarrow$

positive disjointness preserving operator

Theorem (van Imhoff, 2018)

Let P and Q be nonempty compact Hausdorff spaces, let X and Y be order dense subspaces of $\mathrm{C}(P)$ and $\mathrm{C}(Q)$, respectively, and $T\colon X \to Y$ linear.

(i) *If T is a Riesz* homomorphism, then there exist $w\colon Q \to \mathbb{R}_+$ and $\alpha\colon Q \to P$ such that for every $x \in X$ and $q \in Q$ we have*

$$(Tx)(q) = w(q)x(\alpha(q)) \quad (*).$$

If for every $p_1, p_2 \in P$ with $p_1 \neq p_2$ there is $x \in X$ such that $x(p_1) = 0$ and $x(p_2) = 1$, then w can be taken such that w is continuous on Q, the map α is uniquely determined on $\{q \in Q; w(q) > 0\}$, and on this set α is continuous.

(ii) *If for T there exist $w \in \mathrm{C}(Q)$, $w \geq 0$, and $\alpha\colon Q \to P$ continuous on $\{q \in Q; w(q) > 0\}$ such that $(*)$ holds for every $x \in X$ and $q \in Q$, then T is a Riesz* homomorphism.*

Similar characterizations:

$$\begin{array}{c} \text{complete Riesz homomorphism} \quad \checkmark \\ \Downarrow \\ \text{Riesz homomorphism} \quad ? \\ \Downarrow \\ \text{Riesz* homomorphism} \quad \checkmark \\ \Downarrow \\ \text{mild Riesz* homomorphism} \quad ? \\ \Downarrow \\ \text{positive disjointness preserving operator} \quad ? \end{array}$$

Disjointness preserving C_0-semigroups

C_0-semigroups on Banach lattices are well-understood, in particular the relation between properties of the C_0-semigroup and its generator $A\colon x \mapsto \lim_{t\downarrow 0} \frac{T(t)x-x}{t}$.

Theorem (Arendt, 1986)

Let X be a Banach lattice and let $T\colon [0,\infty) \to \mathcal{L}(X)$ be a C_0-semigroup with generator $A\colon X \supseteq \mathcal{D}(A) \to X$ such that, for every $t \in [0,\infty)$, the operator $T(t)$ is disjointness preserving. Then A is band preserving.

What we know:

Theorem (van Gaans, K., Zhang, 2018)

Let (X, X_+) be a directed ordered Banach space with closed cone and a monotone norm and let $T\colon [0,\infty) \to \mathcal{L}(X)$ be a C_0-semigroup with generator $A\colon X \supseteq \mathcal{D}(A) \to X$ such that, for every $t \in [0,\infty)$, the operator $T(t)$ is disjointness preserving. Then A is band preserving.

A large part of the theory of disjointness preserving operators (and dp. C_0-semigroups in Banach lattices) revolves around the following result [Meyer 1976]:

In Archimedean vector lattices, every order bounded disjointness preserving operator has a modulus; in particular, it is regular.

Not true in Archimedean pre-Riesz spaces!
Counterexample in [van Gaans, K., 2019 (book)]; the space there is not weakly pervasive (and, hence, not pervasive).

What we don't know:
Is there a version of Meyer's theorem in pervasive (or other nice) pre-Riesz spaces or certain ordered Banach spaces?

Summary

- still some work to do concerning substructures of pre-Riesz spaces
- operators on pre-Riesz spaces got a nice start, but are still a big playground
- spaces of operators on (pre-)Riesz spaces - work in progress!
- new developments ... JB-algebras ...

Work for the next 20 years!

THANK YOU!

Kalauch, A.; Stennder, J.; van Gaans, O.
Operators in pre-Riesz spaces: moduli and homomorphisms
Positivity 25(5) (2021), 2099-2136

Kalauch, A.; van Gaans, O.
Pre-Riesz Spaces
De Gruyter, ISBN 978-3-11-047539-5 (2019)

M. van Haandel
Completions in Riesz space theory
Ph.D. thesis, University of Nijmegen, 1993.