Semicontinuity properties of Banach lattices

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ICMAT

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Similarly, F is order continuous iff the norm topology on F is weaker than order convergence, i.e. $f_\alpha \xrightarrow{\mathrm{o}} f \Rightarrow f_\alpha \to f$.

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All considered conditions pass down to regular sublattices.

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A Banach lattice F is (r-)monotonically complete iff whenever $D \subset F_+$ is disjoint and such that

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$\mathcal{C}_{ph}(M^*)$ is the space of all continuous positively homogeneous functions on M^*. This is the free AM-space over M.

Free AM-spaces

Let E be a normed space. Let $F := \mathcal{C}_{ph}(\mathrm{B}_{E^*}, \mathrm{w}^*)$ be the space of all weak* continuous positively homogeneous functions on B_{E^*} – the free AM-space over E.

Theorem 8 (B., 2026) TFAE:)

- $F = \mathcal{C}(K)$, *for some* K *(isometrically);* • *F is demicontinuous;*
- $\dim E < \infty$*;* • *F is regular in* $\mathcal{C}(\mathrm{B}_{E^*}, \mathrm{w}^*)$*;* • $\|\cdot\|_L \neq 0$.

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Proposition 6

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- *If M has supremum and infimum, then $\mathcal{C}_{ph}(M^*) \simeq \mathcal{C}(K)$, for some K (isometrically).*

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Proposition 6

- *If* M *has supremum and infimum, then* $\mathcal{C}_{ph}(M^*) \simeq \mathcal{C}(K)$, *for some* K *(isometrically).*
- *If* M *is totally ordered, but no sup and no inf, then* $\|\cdot\|_L \equiv 0$.

THANK YOU!