Order Properties of Free Banach Lattices

Asma Ben Rjeb

University of Tunis El Manar, Tunisia

COSA 2026
Marseille, France

September 14–18, 2026

What is a Banach lattice?

Banach lattice

$$\boxed{\text{Banach lattice} = \text{Banach space} + \text{vector lattice} + \text{lattice norm}}$$

A norm $\|\cdot\|$ is a **lattice norm** if

$$|x| \leq |y| \implies \|x\| \leq \|y\|.$$

Classical examples

$C(K)$ with $\|\cdot\|_\infty$

$\ell^p(I)$, coordinatewise order

Lattice homomorphisms and Banach lattice homomorphisms

Definition: lattice homomorphism

Let L, M be lattices. A map $\phi : L \to M$ is a **lattice homomorphism** if it preserves both lattice operations:

$$\phi(x \vee y) = \phi(x) \vee \phi(y) \quad \text{and} \quad \phi(x \wedge y) = \phi(x) \wedge \phi(y), \qquad x, y \in L.$$

Lattice homomorphisms and Banach lattice homomorphisms

Definition: lattice homomorphism

Let L, M be lattices. A map $\phi : L \to M$ is a **lattice homomorphism** if it preserves both lattice operations:

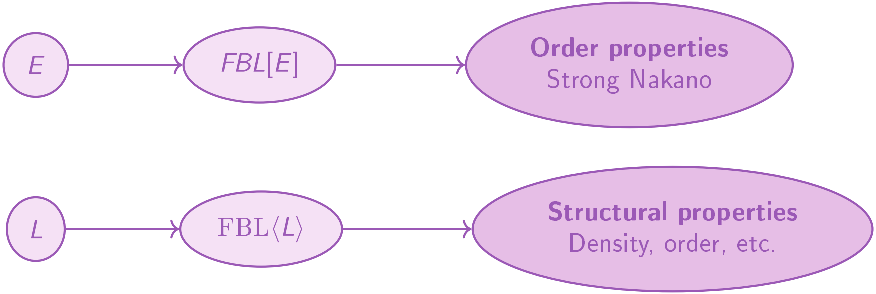
$$\phi(x \vee y) = \phi(x) \vee \phi(y) \quad \text{and} \quad \phi(x \wedge y) = \phi(x) \wedge \phi(y), \qquad x, y \in L.$$

Definition: Banach lattice homomorphism

Let X, Y be Banach lattices. A linear map $T : X \to Y$ is a **(Banach) lattice homomorphism** if

$$T(x \vee y) = Tx \vee Ty \quad \text{and} \quad T(x \wedge y) = Tx \wedge Ty, \qquad x, y \in X.$$

Plan of the talk



The free Banach lattice generated by a Banach space

E

The free Banach lattice generated by a Banach space

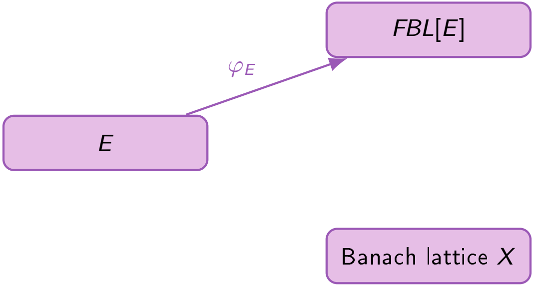
$FBL[E]$

E

The free Banach lattice generated by a Banach space

$$\begin{tikzcd} & FBL[E] \\ E \arrow[ur, "\varphi_E"] & \end{tikzcd}$$

The free Banach lattice generated by a Banach space



The free Banach lattice generated by a Banach space

$$\begin{tikzcd} & FBL[E] \\ E \arrow[ur, "\varphi_E"] \arrow[dr, dashed, "T"'] & \\ & \text{Banach lattice } X \end{tikzcd}$$

The free Banach lattice generated by a Banach space

$$\begin{tikzcd}
 & FBL[E] \arrow[d, dashed, "\exists!\ \widehat{T}"] \\
E \arrow[ur, "\varphi_E"] \arrow[r, dashed, "T"'] & \text{Banach lattice } X
\end{tikzcd}$$

Definition (Avilés, Rodríguez, Tradacete, 2018)

Let E be a Banach space. The **free Banach lattice generated by** E is a Banach lattice $FBL[E]$ together with a linear isometric embedding $\varphi_E : E \to FBL[E]$ such that: for every Banach lattice X and every operator $T : E \to X$ there exists a unique Banach lattice homomorphism $\widehat{T} : FBL[E] \to X$ satisfying

$$\widehat{T} \circ \varphi_E = T \quad \text{and} \quad \|\widehat{T}\| = \|T\|.$$

Existence of $FBL[E]$

Given a Banach space E,
does the free Banach lattice generated by E exist?

Existence of $FBL[E]$

Given a Banach space E,
does the free Banach lattice generated by E exist?

Yes.

Avilés, Rodríguez, Tradacete (2018)

The free Banach lattice generated by a Banach space,
J. Funct. Anal. 274 (2018), 2955–2977.

Construction of $FBL[E]$ (1/2)

Step 1 – the ambient space

$$\boxed{H[E] = \{f : E^* \to \mathbb{R} \ : \ f(tx^*) = t\,f(x^*) \ \text{ for all } t \geq 0,\ x^* \in E^*\}}$$

positively homogeneous functions on E^*.

Construction of $FBL[E]$ (1/2)

Step 1 – the ambient space

$$\boxed{H[E] = \{f : E^* \to \mathbb{R} \ : \ f(tx^*) = t\,f(x^*) \ \text{ for all } t \geq 0,\ x^* \in E^*\}}$$

positively homogeneous functions on E^*.

Step 2 – the free norm

For $f \in H[E]$,

$$\boxed{\|f\|_{FBL[E]} = \sup\left\{\sum_{k=1}^{n} |f(x_k^*)| : n \in \mathbb{N},\ x_1^*, \ldots, x_n^* \in E^*,\ \sup_{x \in B_E} \sum_{k=1}^{n} |x_k^*(x)| \leq 1\right\}.}$$

Construction of $FBL[E]$ (1/2)

Step 1 – the ambient space

$$\boxed{H[E] = \{f : E^* \to \mathbb{R} \ : \ f(tx^*) = t\,f(x^*) \ \text{ for all } t \geq 0,\ x^* \in E^*\}}$$

positively homogeneous functions on E^*.

Step 2 – the free norm

For $f \in H[E]$,

$$\boxed{\|f\|_{FBL[E]} = \sup\left\{\sum_{k=1}^{n} |f(x_k^*)| : n \in \mathbb{N},\ x_1^*, \ldots, x_n^* \in E^*,\ \sup_{x \in B_E} \sum_{k=1}^{n} |x_k^*(x)| \leq 1\right\}.}$$

Step 3 – keep what has finite norm

$H_1[E] = \{f \in H[E] : \|f\|_{FBL[E]} < \infty\}$ is a Banach lattice for the pointwise lattice operations.

Construction of $FBL[E]$ (2/2)

Step 4 – the canonical evaluations

For $x \in E$,

$$\boxed{\delta_x : E^* \longrightarrow \mathbb{R}, \qquad \delta_x(x^*) = x^*(x).}$$

Each $\delta_x \in H_1[E]$ with $\|\delta_x\|_{FBL[E]} = \|x\|_E$: an isometric embedding.

Construction of $FBL[E]$ (2/2)

Step 4 – the canonical evaluations

For $x \in E$,

$$\boxed{\delta_x : E^* \longrightarrow \mathbb{R}, \qquad \delta_x(x^*) = x^*(x).}$$

Each $\delta_x \in H_1[E]$ with $\|\delta_x\|_{FBL[E]} = \|x\|_E$: an isometric embedding.

Definition: $FVL[E]$

$FVL[E] = \mathrm{lat}\{\delta_x : x \in E\}$ – the vector lattice generated by the evaluations δ_x.

Construction of $FBL[E]$ (2/2)

Step 4 – the canonical evaluations

For $x \in E$,

$$\boxed{\delta_x : E^* \longrightarrow \mathbb{R}, \qquad \delta_x(x^*) = x^*(x).}$$

Each $\delta_x \in H_1[E]$ with $\|\delta_x\|_{FBL[E]} = \|x\|_E$: an isometric embedding.

Definition: $FVL[E]$

$FVL[E] = \mathrm{lat}\{\delta_x : x \in E\}$ – the vector lattice generated by the evaluations δ_x.

Step 5 – close up

$$\boxed{FBL[E] = \overline{FVL[E]}^{\|\cdot\|_{FBL[E]}} \subset H_1[E].}$$

From Fatou to Nakano

Let $A \subset X_+$ be upward directed.

Fatou

If $\sup A$ exists,

$$\|\sup A\| = \sup_{a \in A} \|a\|.$$

Nakano

For order-bounded A,

$$\inf_{b \geq A} \|b\| = \sup_{a \in A} \|a\|.$$

From Fatou to Nakano

Let $A \subset X_+$ be upward directed.

Fatou

If $\sup A$ exists,

$$\|\sup A\| = \sup_{a \in A} \|a\|.$$

Nakano

For order-bounded A,

$$\inf_{b \geq A} \|b\| = \sup_{a \in A} \|a\|.$$

- **Wickstead (2007):** X has Nakano $\iff$ its Dedekind completion X^δ has Fatou.

The strong Nakano property

A Banach lattice X has the **strong Nakano property** if for every upwards directed norm bounded set $A \subset X_+$ there exists an upper bound y_0 of A in X such that

$$\|y_0\| = \sup\{\|a\| : a \in A\}.$$

The strong Nakano property

A Banach lattice X has the **strong Nakano property** if for every upwards directed norm bounded set $A \subset X_+$ there exists an upper bound y_0 of A in X such that

$$\|y_0\| = \sup\{\|a\| : a \in A\}.$$

$$\boxed{\text{Strong Nakano} \implies \text{Nakano} \implies \text{Fatou}}$$

The λ-strong Nakano property

Definition (Azouzi, Ben Rjeb, Tradacete, 2025)

- For $\lambda \geq 1$, we say that X has the λ-**strong Nakano property** when every norm-bounded upwards directed set $A \subset X_+$ has an upper bound $b \in X$ with

$$\|b\| \leq \lambda \sup\{\|a\| : a \in A\}.$$

- (λ_+)-**strong Nakano:** for every $\lambda' > \lambda$, there is an upper bound $b \in X_+$ of A with $\|b\| \leq \lambda' \sup_{a \in A} \|a\|$.

The case $\lambda = 1$ is exactly the strong Nakano property.

Y. Azouzi, A. Ben Rjeb, P. Tradacete, *The strong Nakano property in Banach lattices*, RACSAM 119 (2025).

First examples

$C(K)$

For every compact Hausdorff space K, $C(K)$ has the strong Nakano property.

First examples

$C(K)$

For every compact Hausdorff space K, $C(K)$ has the strong Nakano property.

c_0 (Azouzi, Ben Rjeb, Tradacete, 2025)

c_0 has the Nakano property, but does not have the λ-strong Nakano property for any $\lambda \geq 1$.

Y. Azouzi, A. Ben Rjeb, P. Tradacete, *The strong Nakano property in Banach lattices*, RACSAM 119 (2025).

Strong Nakano for free Banach lattices

Avilés, Rodríguez, Tradacete (2018)

For every non-empty set A, the free Banach lattice generated by A, $\mathrm{FBL}(A)$, has the strong Nakano property. Equivalently, since $FBL[\ell_1(A)]$,

$$\boxed{FBL[\ell_1(A)] \text{ has the strong Nakano property.}}$$

Strong Nakano for free Banach lattices

Avilés, Rodríguez, Tradacete (2018)

For every non-empty set A, the free Banach lattice generated by A, $\mathrm{FBL}(A)$, has the strong Nakano property. Equivalently, since $FBL[\ell_1(A)]$,

$$\boxed{FBL[\ell_1(A)] \text{ has the strong Nakano property.}}$$

This is the starting point for our investigation of which Banach spaces E produce $FBL[E]$ with strong Nakano.

Reference: A. Avilés, J. Rodríguez, P. Tradacete, *The free Banach lattice generated by a Banach space*, J. Funct. Anal. 274 (2018), no. 10, 2955–2977.

Two structural shortcuts

AM-spaces (Azouzi, Ben Rjeb, Tradacete, 2025)

If X is an AM-space $\big(\|x \vee y\| = \max\{\|x\|, \|y\|\}\big)$, then

$$\boxed{X \text{ has strong Nakano} \iff X \text{ has a strong unit.}}$$

Two structural shortcuts

AM-spaces (Azouzi, Ben Rjeb, Tradacete, 2025)

If X is an AM-space $\left(\|x \vee y\| = \max\{\|x\|, \|y\|\}\right)$, then

$$\boxed{X \text{ has strong Nakano } \iff X \text{ has a strong unit.}}$$

Monotone completeness (Azouzi, Ben Rjeb, Tradacete, 2025)

If every norm-bounded upward directed subset of X_+ has a supremum in X, then

$$\boxed{\text{Fatou } \iff \text{ Nakano } \iff \text{ strong Nakano.}}$$

Y. Azouzi, A. Ben Rjeb, P. Tradacete, *The strong Nakano property in Banach lattices*, RACSAM 119 (2025).

Directed sets versus sequences

The strong Nakano property concerns arbitrary upward directed sets – but for separable spaces, sequences suffice.

Directed sets versus sequences

The strong Nakano property concerns arbitrary upward directed sets – but for separable spaces, sequences suffice.

Separable case (Azouzi, Ben Rjeb, Tradacete, 2025)

Let X be separable. Then

$$\boxed{X \text{ has strong Nakano} \iff X \text{ has strong } \sigma\text{-Nakano},}$$

i.e. every norm-bounded *increasing sequence* $(x_n) \subset X_+$ has an upper bound b with $\|b\| = \sup_n \|x_n\|$.

Y. Azouzi, A. Ben Rjeb, P. Tradacete, *The strong Nakano property in Banach lattices*, RACSAM 119 (2025).

A positive result: finite dimensions

Theorem (Azouzi, Ben Rjeb, Tradacete, 2025)

There is a **universal constant** λ_{fin}, independent of the dimension, such that for every finite-dimensional Banach space E,

$$\boxed{FBL[E] \text{ has the } (\lambda_{\mathrm{fin}})_+\text{-strong Nakano property.}}$$

Y. Azouzi, A. Ben Rjeb, P. Tradacete, *The strong Nakano property in Banach lattices*, RACSAM 119 (2025).

Two obstructions: c_0 and $L_1[0,1]$

The positive result raises the converse question: what does strong Nakano force on E?

Two obstructions: c_0 and $L_1[0,1]$

The positive result raises the converse question: what does strong Nakano force on E?

Obstruction I (c_0) — Azouzi, Ben Rjeb, Tradacete, 2025

Suppose E is a Banach space such that $FBL[E]$ has the λ-strong Nakano property for some λ. Then E cannot contain a subspace isomorphic to c_0.

$$\boxed{c_0 \not\hookrightarrow E.}$$

Two obstructions: c_0 and $L_1[0,1]$

The positive result raises the converse question: what does strong Nakano force on E?

Obstruction I (c_0) — Azouzi, Ben Rjeb, Tradacete, 2025

Suppose E is a Banach space such that $FBL[E]$ has the λ-strong Nakano property for some λ. Then E cannot contain a subspace isomorphic to c_0.

$$\boxed{c_0 \not\hookrightarrow E.}$$

Obstruction II ($L_1[0,1]$) — Azouzi, Ben Rjeb, Tradacete, 2025

If $FBL[E]$ has the λ-Fatou property for some λ, then E does not contain a closed subspace isomorphic to $L_1[0,1]$.

$$\boxed{L_1[0,1] \not\hookrightarrow E.}$$

Y. Azouzi, A. Ben Rjeb, P. Tradacete, *The strong Nakano property in Banach lattices*, RACSAM 119 (2025).

From Banach spaces to lattices

We studied $E \longrightarrow FBL[E]$: the generator carried only **linear and norm** structure.

$$\text{lattice } L \xrightarrow{\text{construction}} \mathrm{FBL}\langle L\rangle$$

The free Banach lattice generated by a lattice

Definition

Let L be a distributive lattice. The **free Banach lattice generated by** L, denoted $\mathrm{FBL}\langle L\rangle$, is a Banach lattice together with norm bounded lattice homomorphism $\delta_L : L \to \mathrm{FBL}\langle L\rangle$ such that for every Banach lattice X and every norm bounded lattice homomorphism $T : L \to X$, there exists a unique Banach lattice homomorphism $\widehat{T} : \mathrm{FBL}\langle L\rangle \to X$ such that

$$\widehat{T} \circ \delta_L = T \quad \text{with} \quad \|T\| = \sup_{x \in L} \|Tx\|.$$

$$\begin{tikzcd}
 & \mathrm{FBL}\langle L\rangle \arrow[d, dashed, "\exists!\,\widehat{T}"] \\
L \arrow[ur, "\delta_L"] \arrow[r, dashed, "T"'] & X
\end{tikzcd}$$

Two constructions of $\mathrm{FBL}\langle L\rangle$

1. Functional construction

Let

$L^* = \{x^* : L \to [-1,1] : x^* \text{ is a lattice homomorphism}\}$.

For $x \in L$,

$$\delta_x(x^*) = x^*(x).$$

For positively homogeneous $f : L^* \to \mathbb{R}$, define

$$\|f\| = \sup \left\{ \sum_{i=1}^{m} |f(x_i^*)| : \sup_{x\in L} \sum_{i=1}^{m} |x_i^*(x)| \leq 1 \right\}.$$

$\mathrm{FVL}\langle L\rangle = \mathrm{lat}\{\delta_x : x \in L\}$.

Then

$$\boxed{\mathrm{FBL}\langle L\rangle = \overline{\mathrm{FVL}\langle L\rangle}^{\|\cdot\|} \subseteq \mathbb{R}^{L^*}.}$$

Two constructions of $\mathrm{FBL}\langle L\rangle$

1. Functional construction

Let

$L^* = \{x^* : L \to [-1,1] : x^* \text{ is a lattice homomorphism}\}$.

For $x \in L$,

$$\delta_x(x^*) = x^*(x).$$

For positively homogeneous $f : L^* \to \mathbb{R}$, define

$$\|f\| = \sup\left\{\sum_{i=1}^m |f(x_i^*)| : \sup_{x\in L}\sum_{i=1}^m |x_i^*(x)| \leq 1\right\}.$$

$\mathrm{FVL}\langle L\rangle = \mathrm{lat}\{\delta_x : x \in L\}$.

Then

$$\boxed{\mathrm{FBL}\langle L\rangle = \overline{\mathrm{FVL}\langle L\rangle}^{\|\cdot\|} \subseteq \mathbb{R}^{L^*}.}$$

2. Quotient construction

Forget the lattice structure of L and consider the free Banach lattice $\mathrm{FBL}(L)$ generated by the underlying set, with canonical map

$$u : L \longrightarrow \mathrm{FBL}(L).$$

(de Pagter–Wickstead, 2015)

Let I be the closed ideal generated by

$$u(x) \vee u(y) - u(x \vee y),$$

$$u(x) \wedge u(y) - u(x \wedge y), \qquad x, y \in L.$$

Then

$$\boxed{\mathrm{FBL}\langle L\rangle \cong \mathrm{FBL}(L)/I.}$$

What was already known?

Theorem (Avilés–Martínez-Cervantes–Rodríguez Abellán–Rueda Zoca)

Every free Banach lattice generated by a lattice is isomorphic to an AM-space .

A. Avilés, G. Martínez-Cervantes, J.D. Rodríguez Abellán, A. Rueda Zoca, *Lattice embeddings in free Banach lattices over lattices*, Math. Inequal. Appl. 25 (2022), no. 2, 495–509 [Theorem 3.9].

Motivation: comparing the two constructions

	$FBL[E]$	$\mathrm{FBL}\langle L\rangle$
Generator	Banach space E	distributive lattice L
Strong unit	finite-dim. criterion	?
Density	dens $=$ dens(E)	?
Order density of FVL	iff E finite-dim.	?
Quasi-interior point	related to separability of E	?

This table is our motivation

Every question mark below is answered in what follows.

Strong units

Theorem (Ben Rjeb, Tradacete, 2026)

For a distributive lattice L,

$$\boxed{\mathrm{FBL}\langle L\rangle \text{ has a strong unit } \iff L \text{ has a minimum and a maximum.}}$$

Strong units

Theorem (Ben Rjeb, Tradacete, 2026)

For a distributive lattice L,

$$\boxed{\mathrm{FBL}\langle L\rangle \text{ has a strong unit} \iff L \text{ has a minimum and a maximum.}}$$

Corollary (strong Nakano reappears) — Ben Rjeb, Tradacete, 2026

Since $\mathrm{FBL}\langle L\rangle$ is an AM-space,

$$\boxed{\mathrm{FBL}\langle L\rangle \text{ has strong Nakano} \iff L \text{ has a minimum and a maximum.}}$$

A. Ben Rjeb, P. Tradacete, *On the free Banach lattice generated by a lattice*, Banach J. Math. Anal. (2026), article no. 48.

Density character

Definition: density character

$\mathrm{dens}(Y) = \min\{|D| : D \subseteq Y,\ \overline{D} = Y\}.$

Density character

Definition: density character

$\mathrm{dens}(Y) = \min\{|D| : D \subseteq Y, \ \overline{D} = Y\}.$

Theorem (Ben Rjeb, Tradacete, 2026)

For every non-empty distributive lattice L,

$$\boxed{\mathrm{dens}(\mathrm{FBL}\langle L\rangle) = \max\{\aleph_0, |L|\}.}$$

Density character

Definition: density character

$\mathrm{dens}(Y) = \min\{|D| : D \subseteq Y,\ \overline{D} = Y\}$.

Theorem (Ben Rjeb, Tradacete, 2026)

For every non-empty distributive lattice L,

$$\boxed{\mathrm{dens}(\mathrm{FBL}\langle L\rangle) = \max\{\aleph_0, |L|\}.}$$

Interpretation: in particular, $\mathrm{FBL}\langle L\rangle$ is separable iff L is countable – a sharp contrast with $FBL[E]$, where $\mathrm{dens}(FBL[E]) = \mathrm{dens}(E)$.

A. Ben Rjeb, P. Tradacete, *On the free Banach lattice generated by a lattice*, Banach J. Math. Anal. (2026), article no. 48.

Densities of order intervals

For $a \leq b$ in L, write $[a, b]_L = \{x \in L : a \leq x \leq b\}$.

Densities of order intervals

For $a \leq b$ in L, write $[a,b]_L = \{x \in L : a \leq x \leq b\}$.

Theorem (Ben Rjeb, Tradacete, 2026)

For $a < b$ in L,

$$\boxed{\max\{\aleph_0, |[a,b]_L|\} \leq \operatorname{dens}\big([\delta(a),\delta(b)]_{\mathrm{FBL}\langle L\rangle}\big) \leq \max\{\aleph_0, |L|\}.}$$

A. Ben Rjeb, P. Tradacete, *On the free Banach lattice generated by a lattice*, Banach J. Math. Anal. (2026), article no. 48.

Order density of $\mathrm{FVL}\langle L\rangle$

Definition: order dense

$Y \subset X$ is order dense if for every $0 < x \in X_+$ there is $0 < y \in Y$ with $y \leq x$.

Definition: countably order bounded

L is countably order bounded lattice if every countable $A \subseteq L$ has a lower and upper bound in L.

Order density of $\mathrm{FVL}\langle L\rangle$

Definition: order dense

$Y \subset X$ is order dense if for every $0 < x \in X_+$ there is $0 < y \in Y$ with $y \leq x$.

Definition: countably order bounded

L is countably order bounded lattice if every countable $A \subseteq L$ has a lower and upper bound in L.

Theorem (Ben Rjeb, Tradacete, 2026)

If L is distributive and countably order bounded, then $\mathrm{FVL}\langle L\rangle$ is order dense in $\mathrm{FBL}\langle L\rangle$. If L is totally ordered, the converse also holds.

A. Ben Rjeb, P. Tradacete, *On the free Banach lattice generated by a lattice*, Banach J. Math. Anal. (2026), article no. 48.

Projection bands

Theorem (Ben Rjeb, Tradacete, 2026)

If L is distributive and $|L| \geq 2$, the only projection bands of $\mathrm{FBL}\langle L\rangle$ are

$$\boxed{\{0\} \quad \text{and} \quad \mathrm{FBL}\langle L\rangle.}$$

Projection bands

Theorem (Ben Rjeb, Tradacete, 2026)

If L is distributive and $|L| \geq 2$, the only projection bands of $\mathrm{FBL}\langle L\rangle$ are

$$\boxed{\{0\} \quad \text{and} \quad \mathrm{FBL}\langle L\rangle.}$$

As a consequence it is never Dedekind σ-complete, its norm is never order continuous, and it has no atoms.

A. Ben Rjeb, P. Tradacete, *On the free Banach lattice generated by a lattice*, Banach J. Math. Anal. (2026), article no. 48.

Quasi-interior points

Definition: quasi-interior point

$u \in X_+$ is quasi-interior if the principal ideal $X_u = \{x : |x| \leq cu \text{ some } c \geq 0\}$ is dense in X.

Definition: separating subset

$S \subseteq L$ is separating if for every non-zero $\phi \in L^*$ there is $s \in S$ with $\phi(s) \neq 0$.

Quasi-interior points

Definition: quasi-interior point

$u \in X_+$ is quasi-interior if the principal ideal $X_u = \{x : |x| \leq cu \text{ some } c \geq 0\}$ is dense in X.

Definition: separating subset

$S \subseteq L$ is separating if for every non-zero $\phi \in L^*$ there is $s \in S$ with $\phi(s) \neq 0$.

Theorem (Ben Rjeb, Tradacete, 2026)

$$\boxed{\mathrm{FBL}\langle L\rangle \text{ has a quasi-interior point} \iff L \text{ has a countable separating subset.}}$$

A. Ben Rjeb, P. Tradacete, *On the free Banach lattice generated by a lattice*, Banach J. Math. Anal. (2026), article no. 48.

Publications

1. Y. Azouzi, A. Ben Rjeb, P. Tradacete. *The strong Nakano property in Banach lattices.* Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A-Mat. 119 (2025), article no. 99.
2. A. Ben Rjeb, P. Tradacete. *On the free Banach lattice generated by a lattice.* Banach J. Math. Anal. (2026), article no. 48.

Selected references.

- A. Avilés, J. Rodríguez, P. Tradacete. *The free Banach lattice generated by a Banach space.* J. Funct. Anal. 274 (2018).
- A. Avilés, J.D. Rodríguez Abellán. *The free Banach lattice generated by a lattice.* Positivity 23 (2019).
- A. Avilés, G. Martínez-Cervantes, J.D. Rodríguez Abellán, A. Rueda Zoca. *Lattice embeddings in free Banach lattices over lattices.* Math. Inequal. Appl. 25 (2022), no. 2, 495–509.
- B. de Pagter, A.W. Wickstead. *Free and projective Banach lattices.* Proc. Roy. Soc. Edinburgh Sect. A 145 (2015), no. 1, 105–143.
- T. Oikhberg, M.A. Taylor, P. Tradacete, V.G. Troitsky. *Free Banach lattices.* J. Eur. Math. Soc. (2024).
- Y. Azouzi, W. Dhifaoui. *Order denseness in free Banach lattices.* arXiv:2508.11648, 2025.

Thank you

Backup: the free Banach lattice generated by a set

Definition (de Pagter–Wickstead, 2015)

Let A be a set. The **free Banach lattice generated by** A, denoted $FBL(A)$, is a Banach lattice together with a map $\delta_A : A \to FBL(A)$ such that: for every Banach lattice X and every *bounded* map $v : A \to X$ (i.e. $\sup_{a\in A} \|v(a)\| < \infty$, no linearity assumed), there is a unique lattice homomorphism $\widehat{v} : FBL(A) \to X$ satisfying

$$\widehat{v} \circ \delta_A = v \quad \text{and} \quad \|\widehat{v}\| = \sup_{a\in A} \|v(a)\|.$$

$$\begin{tikzcd}
 & FBL(A) \arrow[d, dashed, "\exists!\,\widehat{v}"] \\
A \arrow[ur, "\delta_A"] \arrow[r, dashed, "v"'] & X
\end{tikzcd}$$

Backup: where does λ_{fin} come from? (1/3)

Step 1 — truncated norms

For $f \in H_1[E]$, $k \in \mathbb{N}$,

$$\|f\|_{FBL_k[E]} = \sup\left\{ \textstyle\sum_{i=1}^{k} |f(x_i^*)| : \sup\limits_{x \in B_E} \textstyle\sum_{i=1}^{k} |x_i^*(x)| \leq 1 \right\}, \quad \|f\|_{FBL[E]} = \sup_k \|f\|_{FBL_k[E]}.$$

Backup: where does λ_{fin} come from? (1/3)

Step 1 — truncated norms

For $f \in H_1[E]$, $k \in \mathbb{N}$,

$$\|f\|_{FBL_k[E]} = \sup\Big\{ \textstyle\sum_{i=1}^k |f(x_i^*)| : \sup_{x\in B_E} \sum_{i=1}^k |x_i^*(x)| \le 1 \Big\}, \quad \|f\|_{FBL[E]} = \sup_k \|f\|_{FBL_k[E]}.$$

Szarek's Lemma 6 (Studia Math. 98 (1991), 147–156)

Let $n, N \in \mathbb{N}$, E an n-dimensional Banach space, $A \in \mathcal{L}(\ell_\infty^N, E)$, $\|A\| \le 1$, and weights $(w_i)_{i\le N}$ with $\sum_i w_i = 1$. Then there exist $\sigma \subset \{1, \dots, N\}$ and scalars $(\mu_i)_{i\in\sigma}$ with:

(i) $|\sigma| \le \beta^2 n$;

(ii) $\|\sum_{i\in\sigma} t_i \mu_i A e_i\| \le c^{-1} \max_{i\in\sigma} |t_i|$ for all scalars $(t_i)_{i\in\sigma}$;

(iii) $\sum_{i\in\sigma} w_i \mu_i \ge c$;

where $c > 0$ is a **universal numerical constant**, and $\beta = \sup\{K(v) : v \in \mathcal{L}(\ell_\infty^N, E), \|v\| \le 1\}$ is a K-convexity-type quantity satisfying $\beta \le c_1 (1 + \log n)^{1/2}$.

Backup: where does λ_{fin} come from? (2/3)

Step 3 — consequence (Lemma 3.2)

Applying this with A built from $x_1^*, \ldots, x_N^* \in E^*$ and weights $w_i \propto |f(x_i^*)|$, and combining with the norm formula, gives

$$\boxed{\|f\|_{FBL[E]} \leq \lambda_{\text{fin}} \, \|f\|_{FBL_{k_n}[E]}, \qquad f \in H_1[E],}$$

where λ_{fin} is a fixed function of c alone (e.g. $\sim c^{-2}$) — **not** of $n = \dim E$, even though $k_n = \beta^2 n$ itself does grow with n.

Lemma

Let E be a Banach space of finite dimension. For every $k \in \mathbb{N}$, every $\varepsilon > 0$ and every $f \in H_1[E]_+$ there is $g \in FBL[E]$ such that

1. $f \leq g$,
2. $\|g\|_{FBL_k[E]} \leq (1+\varepsilon)\|f\|_{FBL_k[E]}$.

Y. Azouzi, A. Ben Rjeb, P. Tradacete, *The strong Nakano property in Banach lattices*, RACSAM 119 (2025).

Backup: does λ_{fin} have a numerical value?

This is exactly the open question on the last slide of the main talk: *what is the best universal constant* λ_{fin}*?* *Does the* (1_+)*-strong Nakano property always hold?*