Ruin Theory with investments: good news, really

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Essentials

- Classical collective ruin theory : simple but obsolete (still taught) ; fundamental result exponential decay of ruin probalilities
- The striking feature of models with investments is that the ruin probabilities, as the best, decay as a power function. The rate of decay is a positive root of the cumulant generating function of an increment of the logprice process.
- Proofs of **general results** are based on the theory of distributional equations (called also implicit renewal theory) the approach based on integro-differential equations is also of great interest. Complicated and difficult to explain to students.
- New approach based on easy math : I believe that it is a really a good news.

“Ruin” for semimartingale OU process : old history, 1

- SOU $X = X^u$ is the solution of linear stochastic equation $dX = X_- dR + dP$, $X_0 = u$, where P, R are semimartingales.
- In insurance P is the“business” process, $X = X^u$ is the reserve invested in risky asset with the price $S = \mathcal{E}(R) = e^V$.
- Ruin probability $\Psi(u) := \mathbf{P}(\tau^u < \infty)$, $\tau^u := \inf\{t\colon\ X^u_t \le 0\}$. Survival probability $\Phi(u) := 1 - \Psi(u)$.
- Bachelier 1900, $R = 0$, $P = W$ (Ever first work on stochastic processes ! Ruin probability with finite time horizon...).
- Lundberg 1903, $R = 0$, P is compound Poisson process with drift and exponential jumps. Explicit formula for $\Psi(u)$.
- Cramér 1930 $R = 0$, P is a compound Poisson with drift.
- Sparre Andersen 1957 ($R = 0$, P is compound renewal process with drift. ...

“Ruin” for semimartingale OU process : old history, 2

- Frolova 1993 $R = at + \sigma w$, P is compound Poisson process with drift and with exponential jumps (nice guess !). Method : IDE $\Longrightarrow$ ODE. Accessible but one needs to prove smoothness of the ruin probability.
- Paulsen 1993, ... GOU R and P are independent Lévy processes... Method : Kesten–Goldie implicit renewal theory General but involved. In this line, a number of works (e.g., K. and co-authors). Difficult to include in lecture courses.
- K & Pergamenshchikov 2002 Annuity model. Method : IDE $\Longrightarrow$ ODE.
- Vostrikova & Spielmann 2020 General SOU.
- K & Ellanskaya 2021 Models with parameters switching by a telegraph sygnal.
- K & Pergamenshchikov 2022 Models with switching parameters (or with hidden Markov processes with finite number of states).

“Ruin” for semimartingale OU process : not so old history

- Eberlein & K & Schmidt 2022 Ruin prob abilities for a Sparre Andersen model with investments. Further works with students.
- Antipov & K 2026. Ruin problems with investments on a finite interval : PIDEs and their viscosity solutions.

Example : exit probabilities for the GOU process

For r.v. V_1 the cumulant generating function H (always convex) is $H(q) := \ln \mathbf{E}\, e^{-qV_1}$, $V_1 = \ln S_1$, $S > 0$ is a Lévy process with the Lévy triplet (a, σ^2, Π).

Theorem (K.,Pergamenshchikov 2020)

If H has a root $\beta > 0$, $H(\beta+) < \infty$, and $\Pi_P(|\bar{h}|^{\beta}) < \infty$, then

$$0 < \liminf_{u\to\infty} u^{\beta}\Psi(u) \le \limsup_{u\to\infty} u^{\beta}\Psi(u) < \infty. \qquad (*)$$

If, moreover, $\Pi_P(]-\infty, 0[) = 0$ and the law $\mathcal{L}(V_T)$ is non-arithmetic for some $T > 0$, then $\Psi(u) \sim C_\infty u^{-\beta}$, $C_\infty > 0$.

Thus, for the model with upward jumps we have an exact asymptotic if the distribution of the increment of log-price process is non-arithmetic, i.e. is not concentrated on the set $\mathbb{Z}d := \{\pm nd, n = 0, 1, \dots\}$, $d > 0$.

As in Paulsen, it was used implicit renewal theory but more advanced version

Good news : an easy result for a "simple" model,1

The company continuously invests a fraction κ of its capital reserve in a risky asset whose price S is gBm with parameters a, $\sigma > 0$. The remaining fraction $1-\kappa$ is invested in a bank account with a rate of return r. The process $X = X^u$ describing the evolution of capital reserve is given by the SDE

$$X_t = u + \int_0^t \kappa X_s (a\,ds + \sigma\,dW_s) + \int_0^t r(1-\kappa) X_s\,ds + P_t, \quad (1)$$

where W is a standard Wiener process independent of a compound Poisson process P with linear drift, i.e.

$$P_t = ct - \sum_{i=1}^{N_t} \xi_i,$$

where $c > 0$ is constant, N is a Poisson process independent on W, and $(\xi_i)_{i\geq 1}$ is i.i.d. sequence with d.f. F such that $F(0+) = 0$.

Good news : an easy result for a "simple" model, 2

Foe brevity, we consider the case $r = 0$ and $k = 1$.

Theorem

Suppose that $\gamma := 2a/\sigma^2 > 1$, *F is continuous and* $\mathbf{E}\xi_1^{\gamma-1} < \infty$. *Then there is* $G \in C([0,\infty)) \cap C^2((0,\infty))$ *which is the unique solution of the IDE*

$$\frac{1}{2}\sigma^2 u^2 G''(u) + (au + c)G'(u) + \lambda \int_0^\infty (G(u-y) - G(u))\, dF(y) = 0 \quad (2)$$

such that $G(0+) = \Phi(0+)$ *and* $G(\infty) = 1$. *It coincides with the survival probability* Φ. *Moreover, there is a finite constant* $C_\infty > 0$ *such that* $\Psi(u) \sim C_\infty u^{-\gamma+1}$ *as* $u \to \infty$.

The formulation contains assertions on smoothness of Φ under weakest assumption on F never seen before and assertion on the asymptotic of the ruin probability Ψ usually formulated in terms $\beta := \gamma - 1$.

Fram IDE to IE

Using the integration by parts (the product of $\bar{F}(.)$ and $G(.-u)$)

$$\int_0^u G(u-y)F(dy) + \int_0^u G'(u-y)\bar{F}(y)dy = G(u) - \bar{F}(u)G(0+),$$

where $\bar{F} := 1 - F$, and the abbreviations $\alpha := 2c/\sigma^2$, $\gamma := 2a/\sigma 2$, $\mu := 2\lambda/\sigma^2$ we get that (2) is reduced to the following IDE for $g(u) := G'(u)$:

$$u^2 g'(u) + (\gamma u + \alpha)g(u) = \mu G(0+)\bar{F}(u) + Bg(u), \qquad (3)$$

where

$$Bg(u) := \mu \int_0^u g(u-z)\,\bar{F}(z)\,dz = \mu \int_0^u g(y)\bar{F}(u-y)dy \le \mu u \sup_{y \le u} g(y).$$

The equation (3) is equivalent to

$$\left(u^\gamma e^{-\alpha/u} g(u)\right)' = u^{\gamma-2}e^{-\alpha/u}(\mu G(0+)\bar{F}(u) + Bg(u)).$$

We “solve” (3) using the integrating factor $u^\gamma e^{-\alpha/u}$!

Integral equation in a small neighborhood of zero, 1

Integrating from zero to u we obtain IE for $g(u) = g(u,z)$, where $z := G(0+) \in [0,1]$:

$$g(u) = \mu z u^{-\gamma} e^{\alpha/u} \int_0^u t^{\gamma-2} e^{-\alpha/t} \bar{F}(t) dt + u^{-\gamma} e^{\alpha/u} \int_0^u t^{\gamma-2} e^{-\alpha/t} Bg(t) dt.$$

We find sufficiently small u_0 to apply the Banach fixed point theorem in $C([0,u_0])$ to the above equation of the form $g = g_0 + Tg$. Noting that

$$\int_0^u t^{-2} e^{-\alpha/t} dt = \int_{1/u}^{\infty} e^{-\alpha s} ds = e^{-a/u}/\alpha,$$

and using the relationship $\mu/\alpha = \lambda/c$, we get that

$$g_0(u) \le \mu z e^{\alpha/u} \int_0^u t^{-2} e^{-\alpha/t} dt = \lambda z/c.$$

The function g_0 is continuous on $(0,\infty)$ and, by the L'Hôpitale rule,

$$\lim_{u \downarrow 0} g_0(u) = \mu z \lim_{u \downarrow 0} \frac{u^{\gamma-2} e^{-\alpha/u} \bar{F}(u)}{e^{-\alpha/u}(\gamma u^{\gamma-1} + \alpha u^{\gamma-2})} = \lambda z/c.$$

Hence, $g_0(u)$ is continuous on $[0,\infty)$.

Integral equation in a small neighborhood of zero, 2

Similarly,

$$\lim_{u\downarrow 0} Tg(u) = \lim_{u\downarrow 0} \frac{u^{\gamma-2}e^{-\alpha/u}Bg(u)}{e^{-\alpha/u}(\gamma u^{\gamma-1}+\alpha u^{\gamma-2})} = 0,$$

and $Tg(u)$ is also a continuous function on $[0,\infty)$.
For an arbitrary function $f \in C([0,u],||.||_u)$ we have, using the estimate for $Bg(u)$, that

$$|Tg(u)| \le \mu u||g||_u e^{\alpha/u}\int_0^u t^{-2}e^{-\alpha/t}\,dt \le \frac{\lambda u}{c}||g||_u.$$

That is $||Tf||_{u_0} \le \theta||f||_{u_0}$ for $\theta = \lambda u_0/c$. For $u_0 < c/\lambda$ the operator T is a contraction in $C([0,u_0])$.
We consider the relation $g(u) = g_0(u) + Tg(u)$ as an integral equation in the Banach space $C([0,u_0])$. With our choice of u_0, it has, by the Banach theorem applied to the contraction mapping $g \mapsto g_0 + Tg$, a unique fixed point g_*.
We argue now that g on $(0,u_0]$ is already known and equal to g_*. It remains to build the solution of the IDE for g on $[u_0,\infty)$.

Solution on $[0, \infty)$, 1

Lemma

Suppose that $\mathbf{E}\xi_1^{\gamma-1} < \infty$ and $z > 0$. Then IDE has a unique solution $g^* > 0$ on $[u_0, \infty)$, continuously differentiable on (u_0, ∞) and such that $g^*(u_0) = g_*(u_0)$ and $u^\gamma g^*(u) \to C_\infty$ as $u \to \infty$ where $C_\infty > 0$. The function $\hat{g} = g_* I_{[0,u_0)} + g^* I_{[u_0,\infty)}$ is the unique classical solution on $\mathbb{R}_+$.

Proof. Let $\tilde{g}(u) := u^\gamma g(u)$ and consider on $[u_0, \infty)$ the equation

$$\tilde{g}(u)e^{-\alpha/u} = u_0^\gamma e^{-\alpha/u_0} g_*(u_0) + \int_{u_0}^{u} t^{\gamma-2} e^{-\alpha/t} (\mu z \bar{F}(t) + Bg(t))\, dt.$$

Introducing the function $\Upsilon(u) := u_0^\gamma e^{-\alpha/u_0} g_*(u_0) + \int_{u_0}^{u} t^{\gamma-2} e^{-\alpha/t} \upsilon(t) dt$, where $\upsilon(t) := \mu z \bar{F}(t) + \mu \int_0^{u_0} g_*(y)\bar{F}(t-y)dy$, we get that

$$\tilde{g}(u)e^{-\alpha/u} = \Upsilon(u) + \mu \int_{u_0}^{u} t^{\gamma-2} e^{-\alpha/t} \int_{u_0}^{t} y^{-\gamma} \tilde{g}(y) \bar{F}(t-y)\, dy\, dt.$$

Solution on $[0, \infty)$, 2

Exchanging the order of integrals we arrive to the equation

$$\tilde{g}(u) = e^{\alpha/u}\Upsilon(u) + \int_{u_0}^{u} K(u,y)\tilde{g}(y)\,dy,$$

where

$$K(u,y) := \mu e^{\alpha/u} y^{-\gamma} \int_{y}^{u} t^{\gamma-2} e^{-\alpha/t} \bar{F}(t-y)\,dt.$$

The equation for $\tilde{g}$ is the Volterra integral equation of the second kind. It has on the interval $[u_0, \infty)$ a unique continuous solution $\tilde{g}$. It is differentiable and, and since F is continuous, $\tilde{g}$ is continuously differentiable. The same conclusion holds for the function g.

Limit at infinity, 1

We assume $\mathbf{E}\xi_1^{\gamma-1} < \infty$, that is $\int_{u_0}^{\infty} t^{\gamma-2}\bar{F}(t)dt < \infty$.
Using the estimate $\upsilon(t) \le \mu z\bar{F}(t) + \mu u_0 \|g_*\|_{u_0}\bar{F}(t-u_0)$, we get that

$$\int_{u_0}^{\infty} t^{\gamma-2}e^{-\alpha/t}\upsilon(t)dt < \infty$$

and Υ is a continuous bounded increasing function on $[u_0, \infty)$ with finite limit at infinity. Furthermore,

$$\int_{u_0}^{u} K(u,y)\tilde{g}(y)\,dy \le \mu e^{\alpha/u_0}\int_{u_0}^{u}\tilde{g}(y)y^{-\gamma}\int_{y}^{\infty} t^{\gamma-2}e^{-\alpha/t}\bar{F}(t-y)\,dt\,dy.$$

Thus, $\tilde{g}(u) \le e^{\alpha/u}\Upsilon(u) + \mu e^{\alpha/u_0}\int_{u_0}^{u}\tilde{g}(y)V(y)\,dy$, where
$V(y) := y^{-\gamma}\int_{y}^{\infty} t^{\gamma-2}e^{-\alpha/t}\bar{F}(t-y)dt$. By the Gronwall–Bellman lemma

$$\tilde{g}(u) \le e^{\alpha/u}\Upsilon(u)\exp\left\{\mu e^{\alpha/u_0}\int_{u_0}^{u} V(y)\,dy\right\}.$$

Limit at infinity, 2

Due to assumed existence of the moment of ξ_1, we conclude that V is integrable on $[u_0, \infty)$ and $\tilde{g}$ is a continuous bounded function on $[u_0, \infty)$. Moreover, $\tilde{g}(u)$ tends to a constant as $u \to \infty$. Indeed,

$$\lim_{u \to \infty} K(u, y) = \mu V(y).$$

Thus,

$$\lim_{u \to \infty} \tilde{g}(u) = \mu \Upsilon(\infty) + \mu \int_{u_0}^{\infty} \tilde{g}(y) V(y) dy =: C_\infty.$$

Putting $g^* = u^{-\gamma} \tilde{g}$, we get the result.
Remark. If $G(0+) = 0$, then $g_* \equiv 0$, $\hat{g} \equiv 0$, and $G \equiv G(0+) = 0$. Therefore, for such initial value $G \neq \Phi$, since $\Phi(\infty) = 1$.

The verification lemma

The function $\hat{g}$, the unique smooth solution of the equation (**??**) on $[0, \infty)$, depends on $z = G(0+)$, that is $\hat{g}(u) = \hat{g}(u, z)$,

$$G(u) := z + \int_0^u \hat{g}(s; G(0+))\, ds$$

is a solution of the equation (2) for G. We choose the value $z = G(0+)$ from the equation

$$Gz + \int_0^\infty \hat{g}(s, z)\, ds = 1.$$

As $\hat{g}(s, z)$ is a linear function of z, it has a solution on $(0, 1]$.

Lemma

The solution of PDE G with $G(\infty) = 1$ coincides with Φ (Belkina 2014)

The verification lemma, proof

Apply Itô's formula to $(X^u_{t\wedge\tau})$ where X is the reserve dynamics. The ordinary integrals vanish and we get that $G(X^u_{t\wedge\tau^u}) - G(u) = M_t$, where

$$M_t = \int_0^{t\wedge\tau^u} G'(X^u_{s\wedge\tau^u})\sigma X^u_s dW_s + \int_0^{t\wedge\tau^u}\int (G(X^u_{s-}+x) - G(X^u_{s-})(p-q)(ds,dx).$$

is a bounded martingale starting from zero. Hence, $\mathbf{E}[M_\tau] = 0$ for any stopping time τ. The assumption $\gamma > 1$ (i.e. $\tilde\beta > 0$) ensures that $X^u_t \to \infty$ as $t \to \infty$ (a.s.) on the set $\{\tau^u = \infty\}$, i.e., $X^u_\infty = \infty$ (a.s.) on this set. Indeed, by the stochastic Cauchy formula $X_t = Z_t Y^u_t$, where

$$Z_t := e^{(a-\sigma^2/2)t+\sigma W_t} = e^{\sigma^2 t(\beta/2+(1/\sigma)W_t/t}, \quad Y^u_t := u + \int_0^t Z_s^{-1} dP_s.$$

By the LLN $W_t/t \to 0$ (a.s.) as $t \to \infty$. Hence, Z is exponentially increasing. On the set $\{\tau^u = \infty\}$ the process $Y > 0$. Due to the independence of W and P, its limit not equal to zero.
It follows that

$$G(u) = \mathbf{E}[G(X^u_{\tau^u})I_{\{\tau^u=\infty\}}] + \mathbf{E}[G(X^u_{\tau^u})I_{\{\tau^u<\infty\}}] = G(\infty)\mathbf{P}(\tau^u = \infty) = \Phi(u).$$

References

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