Fundamental concepts in financial mathematics: from traditional approaches to recent advances.

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Modeling a financial market

A financial market is defined by a stochastic basis $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$ where :

- $\Omega \neq \emptyset$ is the set of all possible future states of the market on the time interval $[0,\infty)$ or $[0,T]$, $T>0$.
- $(\mathcal{F}_t)_{t\geq 0}$ is a filtration, i.e. a family of increasing σ-algebras (*) : $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$ if $0 \leq t_1 \leq t_2$.
- P is a probability measure on $\sigma(\vee_{t\geq 0}\mathcal{F}_t)$.

(*)

$$\emptyset, \Omega \in \mathcal{F}_t,$$
$$F_t \in \mathcal{F}_t \Rightarrow \Omega \setminus F_t \in \mathcal{F}_t,$$
$$F_t^n \in \mathcal{F}_t, n = 1,2,\cdots, \Rightarrow \bigcup_{n\geq 1} F_t^n \in \mathcal{F}_t.$$

Modeling a financial market

At any time $t \geq 0$, $\mathcal{F}_t$ describes the cumulated family of events we may observe at time t.

Example of event in the french market if the CAC 40 is supposed to be observed :

$$F_t := \{\omega \in \Omega : \text{ the maximum value of the CAC40 between time 0 and t is less than 8150 points.}\}$$

Also, $\mathcal{F}_t$ is the the cumulated information we are supposed to have.

Modeling the price of a risky asset we observe

In the financial market of consideration, we suppose that there are some risky assets, i.e. assets we may trade (buy or sell) at some prices which are uncertain, not known by advance, because of random fluctuations we do not control.

Therefore, the price of a risky asset that we are supposed to observe through the information $(\mathcal{F}_t)_{t\geq 0}$ we have, is modeled by a stochastic process $S = (S_t)_{t\geq 0}$, i.e. a family of random variables $S_t : \Omega \ni \omega \mapsto S_t(\omega) \in R^+$ such that S_t is $\mathcal{F}_t$-measurable, i.e. observable at any time $t \geq 0$: $S_t^{-1}(B) \in \mathcal{F}_t$, for all $B \in \mathcal{B}(\mathbf{R})$.

As $S_t^{-1}(B) = \{S_t \in B\} := \{\omega \in \Omega : \; S_t(\omega) \in B\} \in \mathcal{F}_t$, it is possible to estimate $P(S_t \in B)$ if P is defined on $\mathcal{F}_t \subseteq \sigma(\vee_{t\geq 0}\mathcal{F}_t)$.

But what is P ? Historical probability measure ?

Dynamics of a risky asset

How to model the future trajectories of the risky asset's price : $t \mapsto S_t(\omega)$ for the various values of $\omega \in \Omega$?

Louis Bachelier is considered as the founder of mathematical finance by using a random walk to model the fluctuations of asset prices in his thesis defensed in 1900.

The next breakthrough came with the Black and Scholes model (and Merton..). The famous paper " The Pricing of Options and Corporate Liabilities" was published in 1973 in "The Journal of Political Economy" (JPE), Vol. 81, No. 3, pp. 637–659.

Dynamics of a risky asset

Definition

A standard real-valued Brownian Motion on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$ *is a stochastic process* $W = (W_t)_{t\geq 0}$ *such that :*

- $W_0 = 0$,
- W_t *is* $\mathcal{F}_t$*-measurable, for all* $t \geq 0$,
- *a.s.*(ω), $t \mapsto W_t(\omega)$ *is continuous*
- $W_{t_2} - W_{t_1}$ *is independent of* $\mathcal{F}_{t_1}$ *for all* $0 \leq t_1 \leq t_2$.
- $W_{t_2} - W_{t_1}$ *is distributed as* $\mathcal{N}(0, t_2 - t_1)$ *for all* $0 \leq t_1 \leq t_2$.

Existence ? Norbert Wiener, 1923.

Dynamics of a risky asset : continuous processes

In the Black and Scholes model, the asset price $S = (S_t)_{t \in [0,T]}$ is given by

$$S_t = S_0 e^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t}, \quad t \in [0, T].$$

In terms of dynamics, S satisfies the stochastic differential equation (SDE) :

$$dS_t = \sigma S_t dW_t + \mu S_t dt, \quad t \in [0, T].$$

Intuitively : $S_{t+dt} - S_t = \sigma S_t (W_{t+dt} - W_t) + \mu S_t dt,\ dt \to 0.$

Mathematically : $S_t = S_0 + \int_0^t \sigma S_u dW_u + \int_0^t \mu S_u du.$

☞ Stochastic integrals, Discretization of diffusion processes via Euler schemes.

More generally, we may consider local volatility (and drift) models, i.e. S satisfies the SDE :

$$dS_t = \sigma(t, S_t)S_t dW_t + \mu(t, S_t)S_t dt, \quad t \in [0, T],$$

where σ, μ are two deterministic functions.
Mathematically : $S_t = S_0 + \int_0^t \sigma(t, S_t)S_u dW_u + \int_0^t \mu(t, S_t)S_u du$.

☞ Existence of a solution S, Uniqueness ?

☞ Other SDE : $(\sigma_t)_{t\in[0,T]}$, $(\mu_t)_{t\in[0,T]}$ are stochastic processes depending on others Brownian Motions.

Dynamics of a risky asset : processes with jumps

The classical way to model a price process with jumps (discontinuous paths in time) is to consider a so called Levy process. A particular case, especially useful in insurance, is defined thanks to a compound Poisson process. For instance :

$$S_t = S_0 e^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t} + \sum_{i=1}^{N_t} Y_i,$$

where W is a B.M, N is a Poisson process and $(Y_i)_{i=1}^{\infty}$ is a family of i.i.d. random variables independent of N.

The trajectories of S are discontinuous at times $T_1 < T_2 < \cdots$ the jump stopping times defining N and the values of the jumps are $Y_1, Y_2, \cdot$.

☞ In insurance, $T_1, T_2, \cdots$ may be the dates of the successive sinisters.

Dynamics of a risky asset : processes with jumps

Processes with jumps are especially considered in actuarial science (for insurance) :

Yuri Kabanov, On exit probabilities for generalized Ornstein–Uhlenbeck processes.

Michèle Vanmaele, Numerical Valuation of European Options under Two-Asset Infinite-Activity Exponential Lévy Models.

Duc Thinh Vu, Optimal Additional Voluntary Contributions in the Presence of Jumps.

Dynamics of a risky asset : calibration and weaknesses

Once the form of the S.D.E. is chosen by the practitioner, it remains to calibrate the parameters, such as volatility and drift, to the real data.

☞ This leads to approximation errors we would like to estimate and control.

☞ Model error : in the real market, there is not a single price per asset but an order book composed of several bid/ask prices and corresponding available quantities. Bid/ask prices may be viewed as mid prices plus transaction costs.

See the talk by :

Mathieu Rosembaum, A unified theory of **order flow**, market impact and **volatility**.

Mihail Zervos, Long-run portfolio optimisation in the presence of **proportional transaction costs** : equivalent risk sensitive and robust formulations.

Amal Omrani, Explicit Characterization and Backward Construction of Superhedging Prices with **Transaction Costs**.

Crista Cuchiero, Dynamic universal approximation and optimal control for path-dependent systems via signature SDEs.

Market Equilibrium Theory

In mathematical economics and, more broadly, in mathematical finance, the concept of market equilibrium is fundamental.

It serves as a key condition for defining and characterizing the "price" of certain financial products, such as derivatives. This framework also extends to problems involving competitive agents, each of whom aims to optimize their own utility function.

We gonna talk about market equilibrium and optimization in the following presentations :

Johannes Langner, $\mathcal{P}$-Sensitive Functions and Localizations.

Emma Hubert, Revisiting contract theory with volatility control.

Paolo Guasoni, Holding Stocks, Trading Bonds.

Laurence Carassus, On the existence of personal equilibria in multistep incomplete financial markets.

Luciano Campi, Optimal Mean Field CCEs ;Optimal Coarse Correlated Equilibria in Mean Field Games : Linear Programming and No-Regret Learning.

Consider a financial market defined by a stochastic basis $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$, a risk-free asset $S_t^0 = 1$ (with a return of 0 for simplicity), and a risky asset described by a stochastic process $S = (S_t)_{t\in[0,T]}$.

Definition

A Call option is a contract between a buyer and a seller. In exchange for an upfront premium (price), the seller commits to delivering the terminal payoff $(S_T - K)^+ = \max(S_T - K, 0)$ to the ***buyer*** *at the future time $T > 0$, where $K > 0$ is a predetermined strike price and S_T is the random price of the asset S at time T.*

What is the **maximal price** the buyer should be willing to pay for this call option ? Conversely, what is the **minimal price** the seller should demand ?

From the seller's perspective, the **minimum initial price** P_0 at time $t = 0$ is defined as the smallest amount of initial capital $V_0 = P_0$ he needs to invest and ensure that, by the terminal date $T > 0$, the accumulated wealth V_T is sufficient to deliver the promised payoff $(S_T - K)^+$, i.e. $V_T \geq (S_T - K)^+$ (super-hedging or replicating) or, maybe, $V_T = (S_T - K)^+$ (hedging or replicating) .

Classical hypothesis : The investment is split between the risk-free asset and the risky asset :

$$V_t = \theta_t^0 S_t^0 + \theta_t S_t, \quad t \in [0, T].$$

Self-financing condition : The only capital contribution is made at the initial time $t = 0$:

$$dV_t = \theta_t^0 dS_t^0 + \theta_t dS_t = \theta_t dS_t,$$

i.e. the variations of the investment $V = (V_t)_{t \geq 0}$ are only due to the variations of the assets as there is no extra capital contribution but the initial one. Also, there is no transaction cost when changing the quantities (θ^0, θ) by selling/buying a fraction of the assets we have invested.

What is the minimal V_0 we need, what is the corresponding strategy (θ^0, θ), to ensure that $V_T \geq (S_T - K)^+$ with probability 1 ?

Arbitrage theory and pricing : traditional approach to solve the problem

Since Black, Scholes and Merton, the classical method consists in supposing a "market equilibrium", we call a No-Arbitrage condition (NA in discrete time, NFL or NFLVR in continuous time) that allows us to mathematically compute a minimal super-hedging price.

NA : There is no possibility for the agents of the financial market to start a self-financing portfolio process $V = (V_t)_{t \in [0,T]}$ allowing to get a non negative and non null gain for sure, i.e. we can not have : $V_T \geq V_0$ a.s. (no risk to face a loss) and $P(V_T > V_0) > 0$ (a possibility of having a positive gain).

Arbitrage theory and pricing : traditional approach to solve the problem

Suppose that $S^0 = 1$.

Theorem (FTAP by Dalang, Morton, Willinger, 1990, in discrete-time)

The NA condition holds if and only if there exists $Q \sim P$ such that S is a martingale under Q.

In continuous time, this is NFLVR and S is a local martingale under Q.

Arbitrage theory and pricing : traditional approach to solve the problem

In discrete time, suppose NA and let $\mathcal{M}$ be the set of all risk neutral probability measures $Q \sim P$ under which S is a martingale.
Consider a payoff $\xi_T \geq 0$, $\mathcal{F}_T$-measurable (for instance $\xi_T = (S_T - K)^+$).
Then, we have :

Theorem

There exists a minimal super-hedging price for the payoff $\xi_T \geq 0$. It is given by :

$$V_0 = \sup_{Q \in \mathcal{M}} E_Q(\xi_T).$$

Arbitrage theory and pricing : traditional approach to solve the problem

When the risk neutral probability measure Q is unique

Definition

We say that the financial market is complete if for any bounded $\mathcal{F}_T$-measurable payoffs ξ_T, there exists a self-financing portfolio process $V = (V_t)_{t\in[0,T]}$ such that $V_T = \xi_T$.

Theorem

Suppose that NA holds. Then, the financial market is complete if and only if there exists a unique risk neutral probability measure Q.

Under this condition, the minimal super-hedging price for the payoff $\xi_T \geq 0$ is actually the unique replicating price V_0, i.e. $V_T = \xi_T$ for some S.F. V, and we have $V_0 = E_Q(\xi_T)$.

Arbitrage theory and pricing : drawbacks and extension of the traditional approach

What happens if we change the information (filtration) we have ? Enlargement of filtration, see the talk by Tahir Choulli, Novel Esscher Concepts for various Risks in Finance and Insurance : Theory and empirical studies.

How to numerically compute $V_0 = \sup_{Q \in \mathcal{M}} E_Q(\xi_T)$ in an incomplete market ? From a theoretical model, from historical data ?

What happens if we have several bid/ask prices or transaction costs for a same risky asset ?

Arbitrage theory and pricing : models with transaction costs

Let (θ^0, θ) be the invested quantities in the assets S^0 and S respectively. The liquidation value $L_t(\theta^0, \theta)$ is no more $V_t = \theta^0_t S^0_t + \theta_t S_t$!

For instance, for a bid/ask model S^b_t, S^a_t, we have :

$$V_t = \theta^0_t S^0_t + (\theta_t)^+ S^b_t - (\theta_t)^- S^a_t.$$

With $S_t = (S^b_t + S^a_t)/2$, $\epsilon_t = (S^a_t - S^b_t)/(2S_t)$, we have $S^b = S_t(1 - \epsilon_t)$, $S^a = S_t(1 + \epsilon_t)$ and

$$V_t = L_t(\theta^0, \theta) = \theta^0_t S^0_t + \theta_t S_t - \epsilon_t |\theta_t| S_t$$

defines a model with proportional transaction costs.

Arbitrage theory and pricing : models with transaction costs in discrete time

We introduce the random solvency set

$$G_t := \{(\theta^0, \theta) : \; L_t(\theta^0, \theta) \geq 0\}.$$

If G_t is closed and $(-1, 0) \notin G_t$,

$$L_t(\theta^0, \theta) = \max\{c : \; (\theta^0, \theta) - (c, 0) \in G_t\}.$$

With $C_t(\theta^0, \theta) = -L_t(-(\theta^0, \theta))$, we have

$$C_t(\theta^0, \theta) = \min\{c : \; (c, 0) - (\theta^0, \theta) \in G_t\}.$$

Arbitrage theory and pricing : models with transaction costs in discrete time

Definition

A portfolio process $V = (\theta^0, \theta)$ *is said self-financing in discrete time* $t = 0, 1, \cdots, T$ *if*

$$\Delta V_t = V_t - V_{t-1} \in -G_t, \text{ a.s.}, \forall t = 1, \cdots, T,$$

i.e. $V_{t-1} = V_t + (-\Delta V_t)$ *where* $C_t(\Delta V_t) \leq 0$.

In continuous-time, the portfolio processes are supposed to be of finite variations and SF condition reads as : $\dot{V}_t = \frac{dV_t}{\|V\|_t} \in -G_t$ a.s. in the Campi-Schachermayer model.

Arbitrage theory and pricing : models with transaction costs in discrete time

The SF condition defines a stochastic pre-order between random variables : $V_{t-1} \gg_{G_t} V_t$ iff $V_{t-1} - V_t \in G_t$, $t = 1, \cdots, T$.

The super-hedging problem reads as $V_T \gg_{G_T} \xi_T$, where ξ_T is a vector-valued payoff.

See papers by Kabanov and Lepinette (Essential supremum and essential maximum with respect to random preference relations, JME) where we introduce a concept of "smallest" random variables w.r.t. a random preorder in order to define the "smallest" SF porfolio processes super-hedging a given payoff ξ_T.

Arbitrage theory and pricing : models with proportional transaction costs in discrete time

With proportional transaction costs, the solvency set is a stochastic convex cone, this is the so-called Kabanov model. In general, we suppose the so-called efficient condition : $G_t \cap (-G_t) = \{0\}$ a.s.

To solve the super-hedging problem $V_T \gg_{G_T} \xi_T$, a no-arbitrage condition NA^r is imposed in the traditional approach :

Theorem

The NA^r condition is equivalent to the existence of Consistent Price Systems, i.e. martingales $Z = (Z_t)_{t\in[0,T]}$ such that $Z_t \in \mathrm{int}\, G_t^*$ a.s. for all t where $G_t^* := \{z \in \mathbf{R}^d : zx \geq 0, \forall x \in G_t\}$.

Arbitrage theory and pricing : models with proportional transaction costs in discrete time

Under NA^r, let $\mathcal{M}$ be the set of all Consistent Price Systems and let us define the set of all super-hedging prices $\Gamma(\xi_T)$, i.e. initial values $V_0 = (\theta_0, \theta)$ of a S.F. portfolio process such that $V_T \gg_{G_T} \xi_T$.

Theorem

We have

$$\Gamma(\xi_T) = \{V_0 : \; V_0 Z_0 \geq E(Z_T V_T), \forall Z \in \mathcal{M}\}.$$

☞ See the papers by Campi, Guasoni, Kabanov, Lepinette, Rasonyi, Schachermayer,...

▲ In practice, this is mainly impossible to compute the possible prices V_0 if we are not able, at least, to identify the CPS Z.

Arbitrage theory and pricing : a new approach without dual elements

In discrete time, knowing that

$$C_t(\theta_t^0,\theta_t)=\theta_t^0+C_t(0,\theta_t)=\theta_t^0+\bar{C}_t(\theta_t),$$

we deduce, with $V=(\theta^0,\theta)$, that the SF condition reads as :

$$\begin{aligned}
\Delta V_t\in -G_t \qquad &\Leftrightarrow C_t(\Delta V_t)\le 0 \Leftrightarrow \Delta\theta_t^0+\bar{C}_t(\Delta\theta_t)\le 0,\\
&\Leftrightarrow \theta_{t-1}^0\ge \theta_t^0+\bar{C}_t(\Delta\theta_t),\\
&\Leftrightarrow \theta_{t-1}^0\ge \operatorname{ess\,sup}_{\mathcal{F}_{t-1}}\left(\theta_t^0+\bar{C}_t(\Delta\theta_t)\right).
\end{aligned}$$

Thus, we must solve the problem backwards for a one-dimensional cash position, which is adjusted according to the risky strategy θ. Here, θ (equivalently $\Delta\theta_t$) serves as the control variable of the problem.

☞ No need of no-arbitrage condition.

Thank you for your attention !

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